One mechanism for the self-organization of nonlinear oscillatory systems is synchronization, as a result of which the interacting subsystems demonstrate a tendency to oscillate at equal (or rationally coupled) frequencies. A "frequency locking" effect occurs in weak interactions, whereas one of the natural frequencies is suppressed in fairly strong interactions. It has been established in physical and numerical experiments that mutual and forced synchronization effects also occur in the interaction of chaotic and stochastic systems.

A stochastic synchronization effect via locking of the mean switching frequency by an external excitation signal was observed in studies of a bistable system driven by an external periodic force and white noise. Stochastic synchronization of switching processes was observed in Ref. 3 in an analysis of two symmetrically coupled bistable systems driven by statistically independent noise sources.

In Refs. 5 and 6, the classical concept of the synchronization of dynamic systems was generalized to the case of symmetrically coupled Lorenz systems functioning in a chaotic regime. The observed synchronization of switching processes induced by "chaos-chaos" intermittence in a Chua circuit was analyzed in Ref. 7.

With this in mind, it seems quite logical to analyze the synchronization process of two symmetrically coupled bistable systems, in which switching is not induced by the action of noise sources, as in Ref. 3, but by the internal dynamics of the subsystems themselves. We selected a Lorenz system as a subsystem possessing these properties.

The dynamic system studied has the form

\[ \begin{align*}
    \dot{x}_1 &= \sigma(y_1 - z_1) + \gamma(x_2 - x_1), \\
    \dot{y}_1 &= r_1 x_1 - x_1 z_1 - y_1, \\
    \dot{z}_1 &= x_1 y_1 - z_1 b, \\
    \dot{x}_2 &= \sigma(y_2 - x_2) + \gamma(x_1 - x_2), \\
    \dot{y}_2 &= r_2 x_2 - x_2 z_2 - y_2, \\
    \dot{z}_2 &= x_2 y_2 - z_2 b.
\end{align*} \]

(1)

It can be seen from Eq. (1) that the interacting subsystems are mismatched in terms of the parameter \( r \). The parameters of the subsystems are such (\( \sigma = 10 \), \( r_1 = 28 \), \( r_2 = 28.8 \), and \( b = 8/3 \)) that a Lorenz attractor exists in each of them. In accordance with the "two states" method, introducing the quantities

\[ \begin{align*}
    x_1' &= \begin{cases} +1, & x_1 > 0 \\ -1, & x_1 < 0 \end{cases}, \\
    x_2' &= \begin{cases} +1, & x_2 > 0 \\ -1, & x_2 < 0 \end{cases},
\end{align*} \]

we shall analyze this model as a system of two symmetrically coupled chaotic bistable systems. The switching process in the subsystems may be characterized by the distribution density of the residence times \( p(\tau) \) in one of the states, denoted by us as +1 and −1. By analogy with a stochastic bistable system, we can introduce the mean transition frequency \( \langle f \rangle \), having determined the switching period as the first moment of a steady-state random process with the distribution density \( p(\tau) \):

\[ \langle T \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^\infty T p(\tau) d\tau, \quad \langle f \rangle = 1/\langle T \rangle. \]

The results of calculations of the switching frequencies \( \langle f_1 \rangle \) and \( \langle f_2 \rangle \) as a function of the degree of coupling are plotted in Fig. 1a. For \( \gamma = 0 \) the switching frequencies of the subsystems are nonidentical, because the values of the parameters \( r_1 \) and \( r_2 \) differ. At this point, it is appropriate to note that increasing the parameter \( r \) of a subsystem causes a slow monotonic increase in its mean switching frequency. With increasing coupling, the mean switching frequencies \( \langle f_1 \rangle \) and \( \langle f_2 \rangle \) initially decrease, reaching a minimum for \( \gamma = 2.8 \). A further increase in coupling causes these frequencies to increase gradually and converge, and at \( \gamma = 6 \) the switching frequencies coincide. It should be noted that the steady-state mean switching frequency in the synchronization regime (\( \gamma > 6 \)) differs from the initial values of \( \langle f_1 \rangle \) and \( \langle f_2 \rangle \) for \( \gamma = 0 \).

By analogy with the classical theory of oscillations, this convergence of the switching frequencies in the subsystems with increasing coupling is logically described as the mutual synchronization of the mean switching frequencies of symmetrically coupled bistable systems.

The increase in the degree of correlation of the switching processes in the subsystems may be illustrated by means of the coherence function:

\[ \Gamma^2(\omega) = \frac{|S_{x_1x_2}(\omega)|}{S_{x_1}(\omega)S_{x_2}(\omega)}. \]
where $S_{x_1x_2}(\omega)$ is the mutual power spectrum of the processes $x_1(t)$ and $x_2(t)$, and $S_{x_1}(\omega)$ and $S_{x_2}(\omega)$ are the spectra of the processes $x_1(t)$ and $x_2(t)$, respectively. It is easy to see that the coherence function varies between 0 and 1. The fact that $\Gamma$ tends to unity in a certain frequency range implies an increase in coherence. The results of the calculations are plotted in Fig. 1b. For zero coupling we find $\Gamma=0$ (see curve 1 in Fig. 1b). A gradual increase in the coupling increases the coherence function in the low-frequency range, which indicates an enhanced degree of correlation between the switching processes in the subsystems (curves 2, 3, 4, 5, and 6 were calculated for $\gamma=0.5, 1.5, 3.0, 4.5$, and 5.5, respectively).

The qualitative changes in the dynamics of the system are caused by bifurcations which take place as the coupling is increased, i.e., symmetry-loss bifurcations, as a result of which an attracting family of saddle cycles is formed in the neighborhood of a symmetric halfspace, as well as Hopf bifurcations of equilibrium states. A detailed analysis of the bifurcation mechanism of the effect is outside the scope of the present article.

It is known from the classical theory of oscillations that the principal characteristic feature of the synchronization effect is the existence of a region of coherent behavior of the subsystems on the coupling–mismatch parameter space. By analogy with the classical case, it seems quite logical to envisage the construction of a similar region for the case where the subsystem is a bistable system with switching processes induced by its complex internal dynamics. The parameter $r$ was selected as the parameter “controlling” the mean switching frequency in the subsystem. In this case, we take the mismatch of the systems to be $\gamma=r_1/r_2$, where $r_1$ and $r_2$ are the parameters of the first and second subsystems, respectively.

Since the time scales characterizing the subsystems in this case are associated with random quantities (these being the mean switching frequencies), we cannot strictly talk of the construction of a clearly defined region on the mismatch–coupling parameter space. In our view, it is more reasonable to envisage the construction of a region within which the switching processes are more coherent (the mean frequencies $\langle f_1 \rangle$ and $\langle f_2 \rangle$ coincide to a given accuracy) than those outside this region. The results are plotted in Fig. 2. The mean switching frequencies $\langle f_1 \rangle$ and $\langle f_2 \rangle$ are synchronized within region $I$, differing by 0.5%. Region 2 is the region of total synchronization of the subsystems ($x_1=x_2, y_1=y_2, z_1=z_2$). It can be seen from the figure that region 2 lies within region $I$, i.e., synchronization of the switching processes in the subsystems with varying coupling precedes the total synchronization of the chaotic oscillations of the subsystems.

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