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REVIEW

Synchronization of Self-Oscillations and Noise-Induced Oscillations

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Abstract—The synchronization of periodic and chaotic oscillations in finite-dimensional dynamical systems is considered on the basis of the classical theory of oscillations. The concepts of the classical theory are applied to nonlinear systems in which oscillations are induced by external noise. The effects of stochastic synchronization and coherence resonance are investigated. It is established that the stochastic synchronization is associated with the locking of the instantaneous phase of chaotic or stochastic oscillations accompanied by the locking of the mean frequency. The results are illustrated by numerical simulation and by radiophysical and medico-biological experiments.

INTRODUCTION

The synchronization of nonlinear oscillations is one of the fundamental phenomena in nature. It can be considered as the simplest example of self-organization of interacting nonlinear systems. By synchronization, one usually means establishing certain relations between the characteristic times, frequencies, or phases of the oscillations in partial systems due to the interaction between these systems. The effect of synchronization, discovered by Huygens [1] as early as the 17th century, was observed in a variety of systems and has found application in different fields of engineering, as is illustrated in many monographs [2–12]. The theory of synchronization has been intensively developed and attracted special interest of scientists in the 20th century in view of the development of electronic communication systems [13]. Presently, the theory of synchronization of periodic self-oscillations [2–23], which has become a classical one, is developed in detail; examples of the synchronization of quasiperiodic oscillations [9, 24, 25] and oscillations in the presence of stochastic forces [9, 26–31] are considered. Within the classical theory, one distinguishes a *forced synchronization*, i.e., the synchronization of oscillations by an external signal, and a *mutual synchronization*, which is observed under the interaction of two self-oscillating systems. Both cases manifest the same phenomena, which are associated with two classical mechanisms of synchronization: the locking of natural frequencies (and, consequently, phases) of oscillations or the suppression of one of the two independent frequencies of quasiperiodic oscillations. The synchronization of self-oscillating systems that occurs only as a result of their interaction is sometimes called a *self-synchronization*, as distinguished from the *controlled synchronization*, which is associated with a purposeful influence on the coupling element [5, 10, 32]. The control of synchronization is related to a wide class of special problems that will not be touched upon in this survey.

The development of synergetics and nonlinear dynamics [33–38] in recent years has given new insight into the role of the classical theory of synchronization. From a modern point of view, regular (periodic and quasiperiodic) oscillating regimes constitute only a small part of the possible types of oscillations. As the number of degrees of freedom of a system increases, nonperiodic oscillations become more and more typical. The accumulation of knowledge on the chaotic dynamics of nonlinear systems has led to the necessity of generalizing the classical concepts of synchronization to the case of chaotic oscillations. Once, there existed a point of view according to which the synchronization and deterministic chaos were considered as opposite trends in the behavior of dynamical systems [10]. Studies based on this viewpoint involved the synchronization of chaos interpreted as a transition from chaotic oscillations to periodic ones under periodic excitation of a chaotic system [39–41]. However, recent investigations have shown that the synchronization can also be observed in the chaotic regime. For instance, the authors of [40] considered, in addition to the transition to the periodic regime, the effect of mutual synchronization of chaotic self-oscillators that manifests itself in the fact that the characteristic peaks in the spectra of partial oscillations coincide. There are three basic types of synchronization of chaotic oscillations described in the literature: *full, frequency-phase, and generalized synchronizations.* The full synchronization of chaos is observed in the interaction of identical chaotic oscillators [42–47]. As the coupling increases, the time realizations of appropriate dynamical variables of the interacting systems completely repeat each other, i.e., the oscillators oscillate in phase. The concepts of full synchronization apply not only to self-oscillating systems but also to the Duffing-type nonautonomous nonlinear oscillators [45], as well as to discrete-time systems (coupled maps) [42, 43]. As the coupling increases, the interaction of systems with slightly mismatched parameters may exhibit a phenomenon similar to the full synchronization [48]. In [49–56], the authors develop the concepts of the frequency–phase synchronization of chaos in self-oscillating systems, that are based on the generalization of the classical concepts of synchronization as the locking or suppression of natural frequencies and the establishment of certain phase relations between the oscillations of interacting oscillators. The concept of generalized synchronization was introduced in [48, 57, 58]. By the generalized synchronization of chaotic oscillations, we mean the establishment of a certain deterministic functional dependence between instantaneous states of partial systems. In interacting chaotic self-oscillating systems with a small mismatch of parameters, one also distinguishes a situation when chaotic oscillations of partial oscillators completely repeat each other with a certain time shift. This type of synchronization of chaos is called a *lag synchronization* [59]. Different types of synchronization of chaos have been considered in the publications of recent years, and various manifestations of synchronization and the associated phenomena have been generalized in the surveys [60, 61].

The synchronization of oscillations manifests itself in ensembles consisting of a large number of interacting oscillators, both periodic [62–66] and chaotic [67–82] ones. The ensembles of oscillators are frequently interpreted as models of a distributed medium; however, in many cases, they may be of interest in and of themselves (for example, in the model of neural networks in biophysics). The synchronization in the ensemble of oscillators is responsible for the limitation of the growth of the attractor dimension and gives rise to stable spatial structures. The possibility of realization of synchronous regimes with different phase shifts is closely related to the phenomenon of multistability, i.e., the coexistence of a set of (regular and chaotic) attractors in the phase space of interacting systems [83]. In turn, the multistability leads to the crises of attractors, the fractalization of attraction basins, etc. Phenomena similar to the synchronization are also observed in the ensembles of bistable elements [84–86]. The investigation of distributed systems described by partial differential equations also reveals the synchronization of space–time structures [10, 87–91].

Recently, phenomena similar to the synchronization have been observed not only in chaotic but also in stochastic systems such as the systems with noise-induced switching [92–97] and the neuron-model-type excitable systems [98–100]. The shape of a signal corresponding to these stochastic processes is very far from sinusoidal, while its power spectrum may miss peaks at certain specific frequencies. However, it turned out that the concept of frequency and phase locking can also be applied to this case. A similar behavior is exhibited by deterministic systems in which the trajectory on a chaotic attractor exhibits irregular jumps in the neighborhood of two equilibrium states. In these systems, just as in stochastic systems, one can observe an effective (in finite time) phase synchronization [94–97].

A comprehensive analysis of synchronization is of great interest due to the development of mathematical modeling and wide application of the methods of nonlinear dynamics in biophysics (neural networks, interacting populations, etc.) [6, 8, 86, 101–103]. In addition, experimental investigations of biological systems for which there does not yet exist a satisfactory mathematical description reveal phenomena close to the known synchronization phenomena [104]. In view of manifold manifestations of the synchronization in a variety of interacting systems, researchers have encountered the problem of generalizing the concepts underlying the classical ideas of synchronization.

In the present survey, we summarize the results of investigating the synchronization phenomena in dynamical systems that are characterized by a finite and relatively small number of degrees of freedom in the absence of noise. We present the classical theory of synchronization of periodic oscillations and analyze the basic types of synchronization of chaotic oscillations. The concept of effective phase synchronization is generalized as applied to stochastic systems with noise-induced phase transitions.

1. SYNCHRONIZATION OF PERIODIC OSCILLATIONS

The Classical Theory of External Synchronization of the Van der Pol Oscillator

A classical example of synchronization is the forced synchronization of a Thomson-type oscillator by a harmonic signal [2–4, 9, 12, 13, 15]. The mathematical model may be given by the nonautonomous Van der Pol equation of the form

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + \omega_0^2 x = a\cos(\omega_1 t + \varphi_0), \qquad (1)$$

where ε is a small positive parameter that characterizes the feedback level; ω_0 is the frequency of periodic oscillations near the oscillation threshold; and a and ω_1 are the amplitude and frequency of the external force, respectively. It is not difficult to obtain a solution to (1) by using a computer. However, to understand the essence of dynamical phenomena, we carry out an analytic analysis, which provides a deeper insight into the physical sense of the synchronization. First, we discuss the concept of the phase of oscillations. The term *phase of oscillations* was first introduced for harmonic processes of the type $x(t) = A\exp(i\omega t) = A(\cos\omega t + i\sin\omega t)$. In polar coordinates, this oscillation is represented as the rotation of the vector A at the constant angular velocity ω. The phase of oscillations is represented by the angle of rotation of the vector A in time, $\varphi = \omega t$. In dissipative nonlinear systems, oscillations cannot be strictly harmonic. What should one do in such a situation? The answer depends on what dynamical system and what type of self-oscillations one deals with.

Let us return to system (1). For small $0 < \varepsilon \ll 1$ and $a = 0$, self-oscillations in the oscillator are close to harmonic ones. Therefore, a solution $x(t)$ to the nonautonomous system (1) can be sought for in the form

$$x(t) = A(t)\cos(\omega_1 t + \varphi(t)),$$
$$\dot{x}(t) = -\omega_1 A(t)\sin(\omega_1 t + \varphi(t)), \qquad (2)$$

where $A(t)$ and $\varphi(t)$ are the oscillation amplitude and phase that slowly vary in time. In other words, we introduce the concepts of an instantaneous amplitude $A(t)$ and instantaneous phase $\varphi(t)$ of the oscillations

$$\Phi(t) = \omega_1 t + \varphi(t). \qquad (3)$$

The "slow" phase $\varphi(t) = \Phi(t) - \omega_1 t$ represents an instantaneous phase difference between the resulting process $x(t)$, which is close to harmonic one, and an external harmonic signal. The slow variation of the phase $\varphi(t)$ in time implies that $\dot{\varphi}(t) \ll \omega_1$. Applying the classical averaging technique, we can readily obtain an equation for the instantaneous amplitude $A(t)$ and instantaneous phase $\varphi(t)$ [2–5, 9, 10, 15]:

$$\dot{A} = \frac{\varepsilon A}{2}\left(1 - \frac{A^2}{4}\right) - \mu\sin\varphi, \quad \dot{\varphi} = \Delta - \frac{\mu}{A}\cos\varphi, \qquad (4)$$

where $\mu = a/2\omega_1$ is a normalized amplitude of the external force and $\Delta = (\omega_0^2 - \omega_1^2)/2\omega_1 \simeq \omega_0 - \omega_1$ is a mismatch between the frequencies of the autonomous oscillator and the external signal in (1).

The system of equations (4) is called a system of "truncated equations." An equilibrium state of system (4) (a stationary point) corresponds to a periodic solution to the original system (1), while a periodic solution to (4) corresponds to a two-frequency quasiperiodic solution of Eq. (1).

Suppose that system (4) has the stationary point $\dot{A} = 0$, $\dot{\varphi} = 0$ as its solution. The condition $\dot{A} = 0$ implies that the oscillation amplitude is time-independent, while the condition $\dot{\varphi} = 0$ implies that $\dot{\Phi} = \omega_1$, i.e., that the frequency of forced oscillations in system (1) coincides with the frequency of the external force. If this oscillating mode is possible and stable, then the frequency of oscillations in the nonautonomous oscillator (1) is changed and proves to be equal to the frequency ω_1 of the external force. The frequency of the oscillator is tuned, thereby implementing the phenomenon of forced synchronization.

Let us consider this phenomenon from the viewpoint of changing the structure of the phase portrait. The phase space of system (4) is a two-dimensional cylinder. Equating the right-hand sides of (4) to zero, we can easily find the coordinates of equilibrium states depicted in Fig. 1a. There are three equilibrium states: a stable node O_1, a saddle point O_2, and an absolutely unstable equilibrium O_3 (a repeller). The stable point O_1 corresponds to the synchronization condition (A = const and φ = const). As the frequency mismatch increases, the equilibrium points O_1 and O_2 start to approach each other, and, for a certain critical mismatch Δ, they merge and vanish as a result of a saddle–node bifurcation (see Fig. 1b). This gives rise to a limit cycle C_1 of the second kind (that encompasses the cylinder), which corresponds to the exit of the system from the synchronization domain and the generation of a quasiperiodic oscillating mode in the original system (1). The domain of the control parameters of system (4) in which the stationary point O_1 remains stable corresponds to the synchronization domain. To determine the boundaries of the synchronization domain, we have to write a linear variational equation for system (4) in the neighborhood of O_1 and find the bifurcation line of the loss of stability for this equilibrium state. This problem can be solved analytically; the relevant results are presented in Fig. 1c.

Within the synchronization domain (domain *I*), the equilibrium state O_1 represents a stable node. On the lines l_a, there occurs a saddle–node tangent bifurcation of the pair of fixed points O_1 and O_2. The line l_c corresponds to the merging and vanishing of the other two equilibrium states O_2 and O_3. Above the line l_c, the equilibrium state O_1 exists and is stable. The points B and C in Fig. 1c, at which the lines l_a and l_c end, represent the cusp points. At these points, all three equilibrium states merge (bifurcation of codimension 2, a triple state of equilibrium). The exit from domain *I* through line l_a corresponds to the birth bifurcation of a second-kind limit cycle: here, a two-frequency oscillating regime arises.

Now, we consider the situation in domain *II* above the bifurcation line l_c, where the only equilibrium state O_1 exists (Fig. 2a). Let us fix the parameters μ and ε and increase the mismatch parameter Δ. For a certain critical value of Δ, the equilibrium state looses stability as a result of a soft Andronov–Hopf bifurcation. A stable limit cycle C_2 (a cycle of the first kind lying on the surface of the cylinder) arises in the neighborhood of O_1. The synchronization regime is broken, and quasiperiodic oscillations arise in the original system (Fig. 2b). A further increase in the mismatch leads to the bifurcational change of the quasiperiodic regime: as a result of a nonlocal homoclinic bifurcation (crisis), the cycle C_2 vanishes. This gives rise to the second-kind cycle C_1 (Fig. 2c). The Andronov–Hopf bifurcation of the equilibrium state O_1 above the line l_c corresponds to the line l_b in Fig. 1c. The lines l_d correspond to the crisis of the cycle C_2 and the birth of the cycle C_1. Thus, the domain

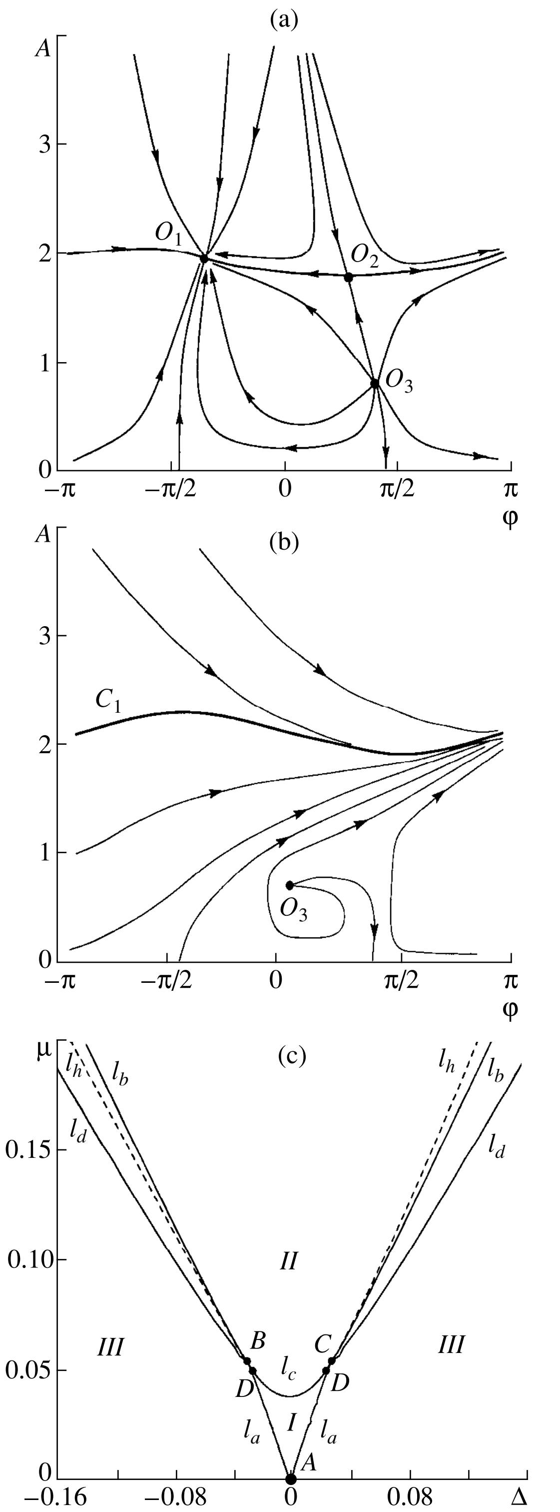

Fig. 1. Phase portraits of system (4) with $\varepsilon = 0.$ and $\mu = 0.052$ for (a) $\Delta = 0.026$ and (b) $\Delta = 0.032$; (c) bifurcation lines of systems (4) and (1) on the plane of parameters Δ and μ.

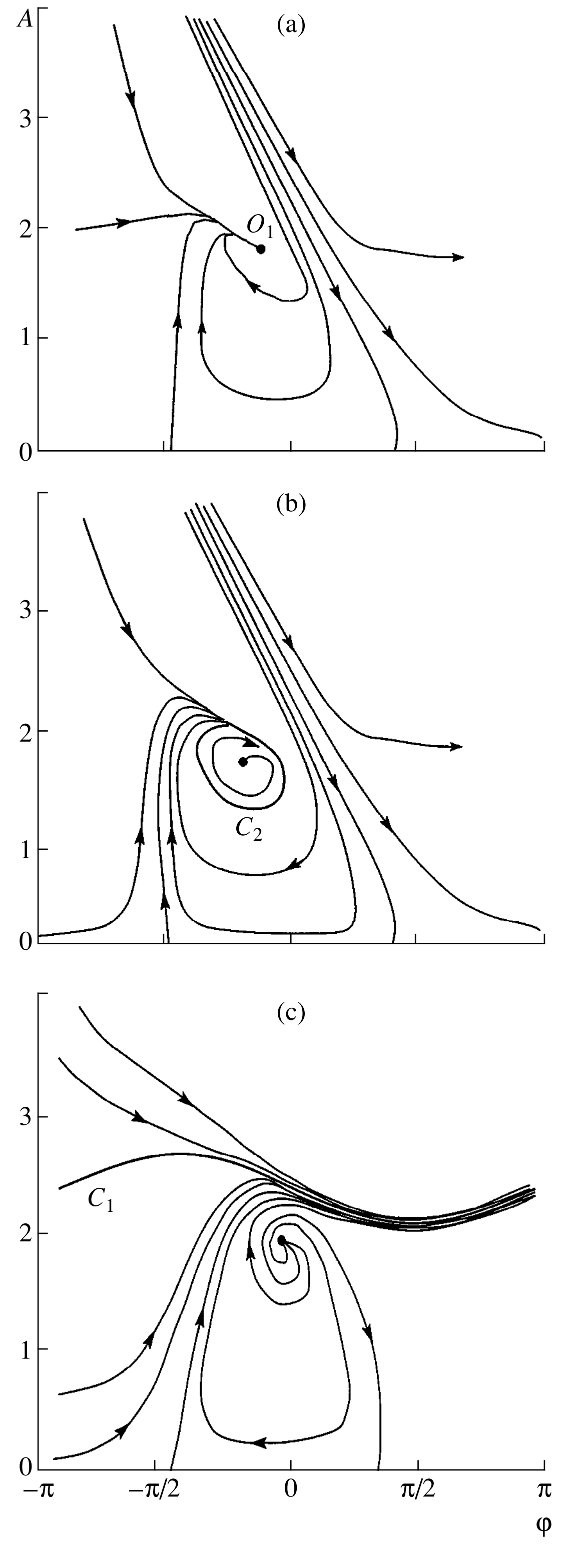

Fig. 2. Phase portraits of system (4) with $\varepsilon = 0.$ and $\mu = 0.056$ for (a) $\Delta = 0.028$, (b) $\Delta = 0.031$, and (c) $\Delta = 0.033$.

of synchronization of the Van der Pol oscillator in Fig. 1c corresponds to domain *I* bounded by the bifurcation lines l_a below the points *B* and *C* and domain *II* between the lines l_b above these points. Inside the domain of synchronization in the original system (1), we have a stable limit cycle whose frequency coincides with the frequency of external force; i.e., the condition $\omega_0/\omega_1 = 1$ is satisfied.

Let us turn again to Fig. 1c and note the following important feature of the phase-locking domain (domain *I*). It is clear from Fig. 1a that the unstable separatrices of the saddle point O_2 are closed on the cylinder, thus forming a stable node O_1. There is an invariant closed curve l in the system that encompasses the cylinder and on which a saddle and a node are situated. This curve represents an image of a resonant torus in system (1). When the saddle point and node merge on the line l_a, the resonant curve is reconstructed giving rise to an ergodic invariant curve (cycle C_1), which represents an image of a two-dimensional ergodic torus. Thus, under small-amplitude external signals, the synchronization corresponds to the resonance on a two-dimensional torus (domain *I*). The torus is destroyed on the line l_c (the invariant curve vanishes); however, the stationary point O_1 exists and is stable. In this case, the synchronization is not connected with the resonance on the torus any longer. The points D (Fig. 1c) at which the lines l_a of saddle–node bifurcations and the lines l_b of Andronov–Hopf bifurcations meet are of special interest. These points are called Bogdanov–Takens bifurcation points [105, 106]. In the neighborhood of these points, a cusp arises on the bifurcation diagram. The synchronization domain *II* always lies above the point D. The realization of the synchronization regime in this domain corresponds to the intersection of the bifurcation lines l_b in the directions from domain *III* toward the interior of domain *II*. In this case, quasiperiodic oscillations in system (1) slowly disappear, and a limit-cycle regime arises. This mechanism is called a synchronization through the asynchronous suppression of oscillations. In terms of truncated equations (4), this mechanism corresponds to the suppression of periodic oscillations of amplitude $A(t)$ and the establishment of a regime with $A = \text{const}$.

It is quite clear that the approximate equations (4) cannot precisely describe the phenomenon of synchronization. For comparison, we carry out a numerical simulation of the original system (1). Since the synchronization regime corresponds to a limit cycle of frequency ω_1, we determine this cycle by setting $\Delta = 0$. Next, varying Δ, we seek the bifurcations of the loss of stability by this cycle. The results of calculations are shown in Fig. 1c (bifurcation lines l_h). One can see that, in the domain of a small-amplitude external signal ($\mu \le 0.05$ in domain *I*), the results of calculations for the original system and for the truncated equations completely coincide.

The Bogdanov–Takens bifurcation points (points D) are also observed in the full system (1). Above the points D, the lines l_h (the lines of the birth of a torus or the lines of Neimark bifurcations) do not coincide any longer with the corresponding lines l_b calculated by the truncated system (4). The greater the parameter μ, the larger is the quantitative discrepancy. However, what is important here is the fact that the main bifurcations inside and on the boundaries of the synchronization domains *I* and *II* in the full system (1) and in system (4) qualitatively coincide. Moreover, investigations have confirmed that these bifurcations are typical in the case of the synchronization of any periodic self-oscillators by an external harmonic force of frequency $\omega_1 = \omega_0 + \Delta$.

It is interesting to know what will happen as the mismatch will further increase, i.e., as the driving frequency will increase in the domain of quasiperiodic oscillations. To answer this question, we introduce the so-called Poincaré rotation number as the ratio of frequencies $\theta = \omega_1/\omega$. Here, ω_1 is the driving frequency and ω is the frequency of the oscillator. In the case of the synchronization at the fundamental mode considered above, we have $\omega = \omega_1$, and the rotation number is equal to unity.

At the exit from the synchronization domain, $\omega \ne \omega_1$, the rotation number will be changed in magnitude and consecutively assume irrational and rational values. In this case, the structure of phase trajectories on the torus will be subject to bifurcational changes. Irrational θ correspond to the ergodic motion of phase trajectories. Rational θ correspond to resonant limit cycles, closed periodic trajectories lying on the surface of a two-dimensional torus. Each resonant cycle exists in a finite range of Δ. These are synchronization domains at the frequencies that represent linear combinations of ω_0 and ω_1.

Qualitatively, the synchronization domains for system (1) are illustrated in Fig. 3a. Within each such domain, the rotation number is invariant, which means the locking of the oscillator frequency by an external signal. Since $\theta = m : n$, as ω_1 varies, the frequency ω will follow the driving frequency so that their ratio remains fixed and equal to the rotation number. The most general and simple model illustrating the effects of external synchronization is given by a one-dimensional map of a circle [107]:

$$x_{n+1} = x_n + \delta + \frac{K}{2\pi}\sin 2\pi x_n, \quad \mathrm{mod}1. \tag{5}$$

The calculated rotation number θ versus mismatch δ is illustrated in Fig. 3b.

Since the rational and irrational numbers are dense everywhere on a straight line, the function $\theta(\delta)$ has a complicated fractal structure. The latter fact was responsible for the term of "devil's staircase" adopted for this function [107]. Thus, if the driving frequency varies within a wide range, one can observe the phenomena of external synchronization that correspond to the general case of resonances m: n, m, n = 1, 2, …. Note that, in the case of a nonautonomous oscillator (1), the axis $a = 0$ in Fig. 3a serves as a bifurcation line of the birth of a torus. The resonance condition, which represents the second bifurcation condition, distinguishes the points with rational values of the rotation number on this line. It is these points on which the synchronization domains, called Arnold tongues, rest. These points characterize a bifurcation of codimension

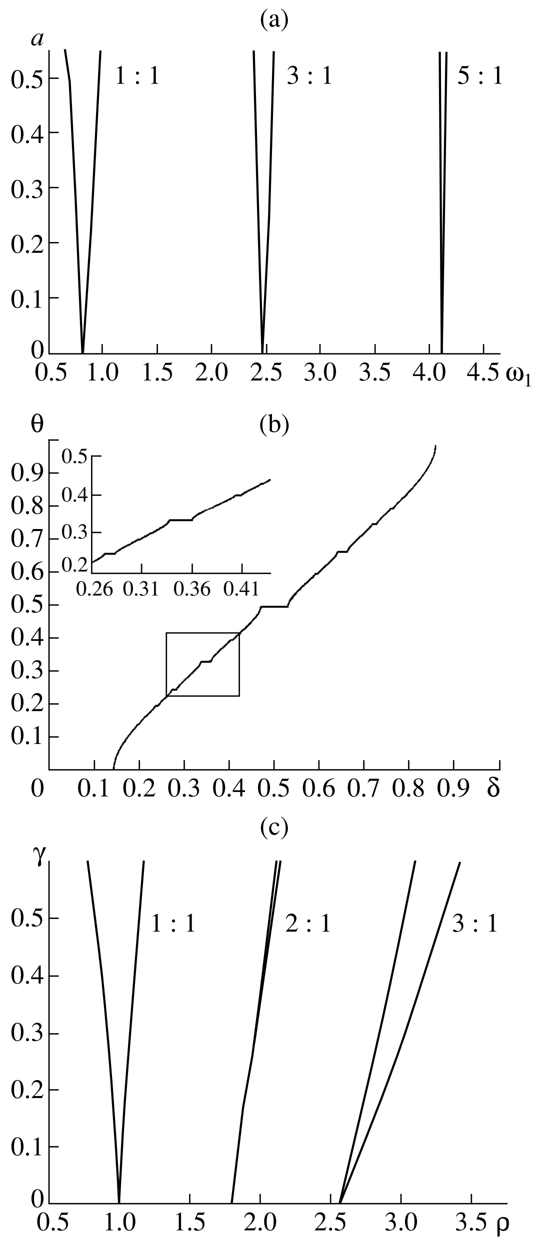

Fig. 3. (a) Synchronization domains of system (1) with rotation numbers of $\theta = 1:1$, $3:1$, and $5:1$; (b) rotation number θ versus parameter δ in map (5) for $K = 0.9$; and (c) synchronization domains on the plane of parameters γ and p in system (6).

two, which requires a two-parameter approach to the analyses.

Mutual Synchronization

The situation considered above corresponds to the external, or forced, synchronization of oscillations. Under the external synchronization, the action on the oscillator is unidirectional. A more general case is the interaction of two self-oscillating systems with different natural frequencies ω_{01} and ω_{02} [16, 18]. During the interaction, the oscillators influence each other since the coupling between them is realized in both directions.

As an example, we analyze two symmetrically coupled Van der Pol oscillators:

$$\begin{aligned} \ddot{x}_1 - \varepsilon(1 - x_1^2)\dot{x}_1 + \omega_{01}x_1 &= \gamma(x_2 - x_1), \\ \ddot{x}_2 - \varepsilon(1 - x_2^2)\dot{x}_2 + \omega_{02}x_2 &= \gamma(x_1 - x_2), \end{aligned} \tag{6}$$

where γ is a coupling coefficient.

Figure 3c represents the structure of the bifurcation diagram of the system of coupled oscillators (6), which is topologically equivalent to the case of a nonautonomous oscillator (see Fig. 3a). From the viewpoint of bifurcation phenomena, the mutual synchronization of oscillations completely coincides with the external synchronization regime described above.

The dynamical phenomena in the nonautonomous (1) and coupled (6) oscillators considered above allow one to formulate the criteria and basic properties of the nonlinear phenomenon called a synchronization.

The basic feature of both the external and mutual synchronizations is the establishment of the oscillating regime with a constant and rational value of the Poincaré rotation number $\theta = m:n$, which is preserved in a certain finite range of parameters of the system, called the domain of synchronization. This domain is characterized by the locking of the frequency and phase of oscillations. The frequency locking implies that there exists a rational ratio of two initially independent frequencies, $\omega_1/\omega_2 = m:n$ everywhere in the domain of synchronization. The phase locking corresponds to a constant phase difference $\varphi(t) = \Phi_1(t) - \Phi_2(t)$ between the oscillations of interacting oscillators in the synchronization domain ($\dot{\varphi} = 0$ and $\varphi_{st} = \text{const}$).

From the physical point of view, the effect of synchronization consists in the fact that two characteristic intrinsic time scales of interacting oscillating systems, which were independent in the absence of interaction, prove to be integral multiples of each other, or rationally related, under the interaction. Here, it is important that this multiplicity proves to be fixed in a certain finite range of parameters of the system, which is called a synchronization domain.

Synchronization of Periodic Oscillations in the Presence of Noise: Effective Synchronization

Real systems always contain noise in the form of natural fluctuations associated with dissipation as well as in the form of random forces of the external environment. It is necessary to find out what fundamental differences in the dynamics of the oscillator can be attributed to random perturbations, in particular, what is the effect of noise on the synchronization.

The problem of the synchronization of a Van der Pol oscillator in the presence of noise was first formulated and solved by R.L. Stratonovich, A.N. Malakhov, and other authors. [26–31]. They considered the effect of noise on an oscillator under the condition that the noise power was much lower than the power of a harmonic signal. In addition, it was assumed that the statistical characteristics of the fluctuations are similar to those of white noise. In particular, the noise source was assumed to be wideband, and the correlation time of noise was assumed to be much less than the relaxation time of stationary values of the amplitude and phase of oscillations. Suppose that the initial stochastic differential equation (SDE) of the oscillator is given by

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + \omega_0^2 x = a\cos(\omega_1 t) + \sqrt{2D_0}\xi(t), \quad (7)$$

where $\xi(t)$ is the δ-correlated Gaussian noise with zero mean: $\langle\xi(t)\rangle \equiv 0$; $\langle\xi(t)\xi(t + \tau)\rangle = \delta(\tau)$. The quantity D_0 characterizes the intensity of noise. Following the arguments of Stratonovich [26], we can obtain truncated SDEs for the instantaneous amplitude $A(t)$ and phase $\varphi(t)$ of oscillations:

$$\begin{aligned} \dot{A} &= \frac{\varepsilon A}{2}\left(1 - \frac{A^2}{4}\right) - \mu\sin\varphi + \frac{D}{A} + \sqrt{2D}\xi_1(t), \\ \dot{\varphi} &= \Delta - \frac{\mu}{A}\cos\varphi + \frac{\sqrt{2D}}{A}\xi_2(t), \end{aligned} \quad (8)$$

where the phase φ represents the instantaneous phase difference between the oscillations $x(t)$ and a harmonic signal, $\mu = a/2\omega_1$, and $D = D_0/2\omega_1^2$. To a certain approximation, the random signals ξ_1 and ξ_2 can be considered as independent δ-correlated normal random functions with zero means: $\langle\xi_{1,2}(t)\rangle \equiv 0$, $\langle\xi_{1,2}(t)\xi_{1,2}(t + \tau)\rangle = \delta(\tau)$. The quantity D characterizes the intensities of the signals ξ_1 and ξ_2. It was noted in [26] that a similar system of equations can be obtained not only under a harmonic external signal, but also when the external signal is a periodic function of any type, for example, when it is a pulse train. When $a \ll \varepsilon$ and $D \ll \varepsilon$, the deviations of the amplitude from its unperturbed value $A_0 = 2$ are small. In this case, one can replace $A(t)$ by the constant value A_0 in the equation for the phase and rewrite this equation as

$$\dot{\varphi} = \Delta - \Delta_s\cos\varphi + \frac{\sqrt{2D}}{A_0}\xi_2(t), \quad (9)$$

where $\Delta_s = \mu/A_0$ is the synchronization bandwidth. Equation (9) describes a Brownian motion of a "particle" with coordinate φ in the one-dimensional tilted periodic potential $U(\varphi) = -\Delta\varphi - \Delta_s\sin\varphi$ (Fig. 4a). The mismatch Δ determines the tilt of the potential and Δ_s determines the height of potential barriers. In the absence of noise ($D = 0$) and when $\Delta < \Delta_s$, the minima $\varphi_k = \arccos(\Delta/\Delta_s) + 2\pi k$ of the potential correspond to the synchronization, since the instantaneous phase difference remains time-independent. The presence of noise leads to the diffusion of the phase difference in the potential $U(\varphi)$: $\varphi(t)$ fluctuates around the minima φ_k of the potential and performs random transitions from one potential well to another, changing abruptly by 2π.

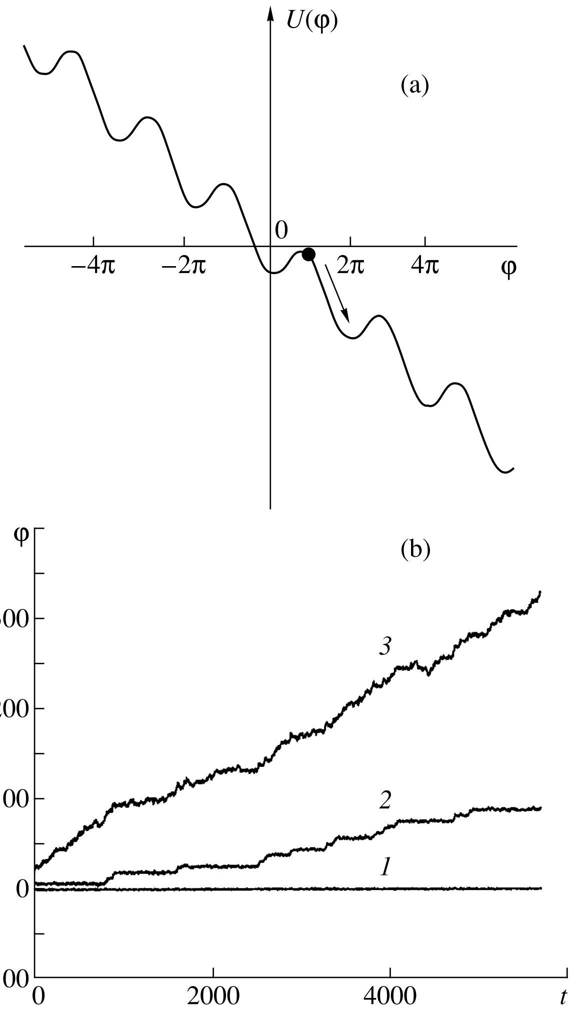

Fig. 4. (a) Periodic potential function $U(\varphi)$ for Eq. (9) and (b) function $\varphi(t)$ for various values of noise intensity D: (*1*) $D = 0.02$, (*2*) $D = 0.07$, and (*3*) $D = 0.22$; $\Delta = 0.06$ and $\varepsilon = 0.15$.

Figure 4b shows the realizations of phase difference for various values of the noise intensity D that are obtained by the numerical integration of SDE (9). One can see that, for a low intensity of noise ($D = 0.02$), the instantaneous phase difference remains close to zero during a long period of time. An increase in the noise intensity results in a decrease in the time φ during which the system stays in one of the potential wells; thus, the jumps of the instantaneous phase difference between different metastable states become manifest ($D = 0.07$). The regions with approximately equal values of the instantaneous phase difference ($\varphi \simeq$ const) are sufficiently clearly manifested; however, the mean value of φ increases with time. The larger the tilt (mis-

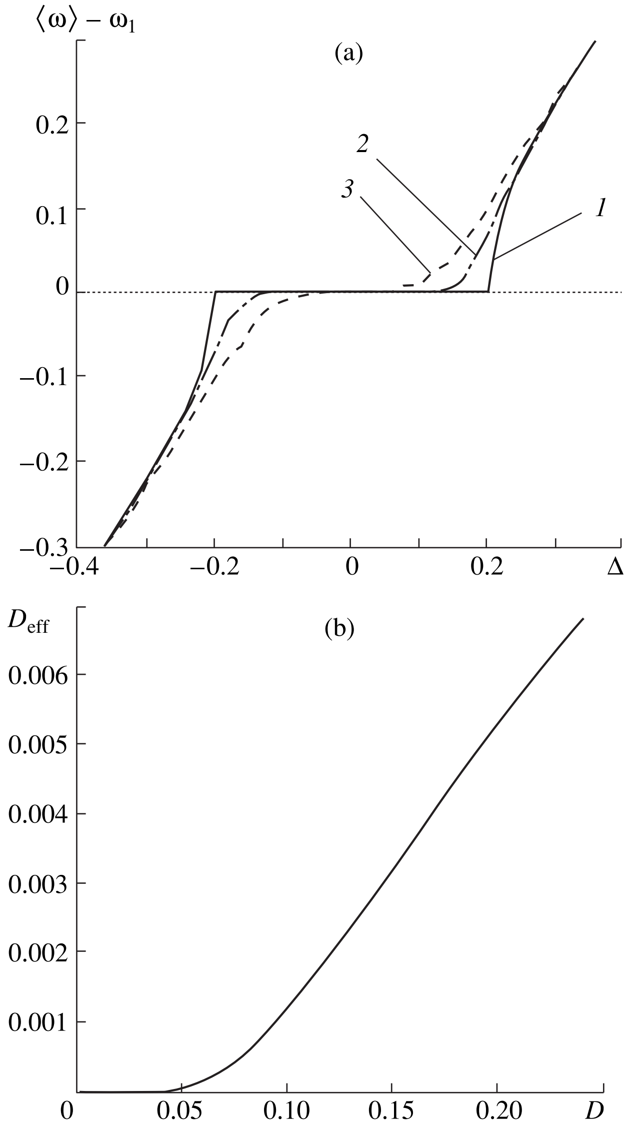

Fig. 5. (a) The difference between the mean oscillation frequency calculated by (9) and the driving frequency as a function of the mismatch parameter for various values of noise intensity D: (*1*) $D = 0$, (*2*) $D = 0.02$, and (*3*) $D = 0.07$; and (b) the effective diffusion coefficient D_{eff} calculated by formula (12) as a function of noise intensity under the assumption of zero mismatch.

match), the shallower the well (the less the amplitude of the driving signal), and the higher the intensity of noise, the smaller is the time period during which the phases of the oscillator and external signal are locked. An increase in the noise intensity results in an unlimited increase in the absolute value of the phase difference (see the functions $\varphi(t)$ for $D = 0.07$ and 0.22) and a variation in the mean frequency of oscillations.

The mean oscillation frequency $\langle\omega\rangle$ can be determined from (9) by averaging $\dot{\varphi}$ over the stationary probability density of phase φ defined on the interval $[0, 2\pi]$:

$$\langle\omega\rangle = \omega_1 + \langle\dot{\varphi}\rangle = \omega_1 + \int_0^{2\pi}(\Delta - \Delta_s\cos\varphi)q(\varphi)d\varphi, \quad (10)$$

where ω_1 is the driving frequency and the stationary probability density $q(\varphi)$ is given by the following equality:

$$q(\varphi) = N\exp\left(\frac{\Delta\varphi - \Delta_s\sin(\varphi)}{Q}\right) \times \int_{\varphi}^{\varphi+2\pi}\exp\left(-\frac{\Delta\psi - \Delta_s\sin\psi}{Q}\right)d\psi, \quad (11)$$

where N is a normalization constant and $Q = D/A_0^2$. The difference between the mean frequency and the driving frequency versus the mismatch parameter is shown in Fig. 5a (curve *1*) in the absence of noise and (curves *2* and *3*) for various values of the noise intensity D.

Under the influence of noise, curves *2* and *3* differ from curve *1* corresponding to $D = 0$. For a small noise level, $D = 0.02$, one can speak of a synchronization in a certain finite range of mismatch parameters (in our particular case, $-0.15 < \Delta < 0.15$). The mean frequency of oscillations virtually coincides with the driving frequency ω_1. As we have already mentioned above, a further increase in the noise intensity leads to the irreversible diffusion of the phase of the synchronized oscillator, thus changing the mean frequency and narrowing down the synchronization domain.

For a dynamical system with noise, one needs a more detailed definition of the concept of synchronization that would be based on the statistical characteristics of oscillations of the nonautonomous oscillator (7). In Malakhov's work [27], it was demonstrated that the concept of synchronization of a Van der Pol oscillator in the presence of noise can be defined either by imposing certain constraints on the signal-to-noise ratio or on the basis of the analysis of phase or frequency fluctuations. Moreover, since the locking time cannot be infinitely large (as it is in a purely dynamical case) in the presence of noise, it makes sense to speak only of the effective synchronization. We introduce the concept of effective synchronization on the basis of the analysis of phase fluctuations.

As we have already mentioned, the presence of noise leads to the diffusion of the instantaneous phase difference; therefore, the definition of the effective synchronization by imposing a constraint on the diffusion rate seems to be quite natural. To determine the diffusion rate of the phase-difference distribution, the following characteristic was suggested in [26]: the effective diffusion coefficient of instantaneous phase difference

$$D_{\text{eff}} = \frac{1}{2}\frac{d}{dt}[\langle\varphi^2(t)\rangle - \langle\varphi(t)\rangle^2]. \quad (12)$$

This parameter indicates how many 2π-jumps makes the instantaneous phase difference in unit time; thus, it is related to the mean period of time during which the phase locking occurs. In a weak-noise approximation, the diffusion coefficient for a Van der Pol oscillator is defined as [26]

$$D_{\text{eff}} = \frac{1}{2\pi}\sqrt{\Delta_s^2-\Delta^2}\left[1+\exp\left(-\frac{2\pi\Delta}{Q}\right)\right] \times \exp\left[-\frac{2}{Q}\left(\sqrt{\Delta_s^2-\Delta^2}-\Delta\arcsin(\Delta/\Delta_s)\right)\right]. \quad (13)$$

The dependence of D_{eff} on the noise intensity for a zero mismatch is shown in Fig. 5b. One can see that an increase in the noise intensity leads to a growth in the diffusion coefficient, which implies a decrease in the time intervals during which the oscillator is synchronized by an external signal. In [27], Malakhov introduced the concept of *effective synchronization.* System (7) is assumed to be effectively synchronized if the mean period of phase locking is much greater than the period of an external harmonic force, i.e., if the following condition holds: $D_{\text{eff}} \le 2\pi\omega_1/n$, where $n \gg 1$ is the number of periods of the external signal during which the oscillator phase remains locked. This criterion for the effective synchronization is not unique since n is actually chosen arbitrarily.

Instantaneous Amplitude and Phase of Quasiperiodic and Aperiodic Oscillations

When oscillations are nonharmonic and even nonperiodic, one faces a serious problem associated with the correct definition of the concepts of instantaneous amplitude and phase.

To introduce the most general definition of the instantaneous phase of an arbitrary process $x(t)$, we avail ourselves of the concept of analytic signal [108–110]. An analytic signal $w(t)$ is a complex function of time, defined as follows.

$$w(t) = x(t)+iy(t) = a(t)\exp[i\Phi(t)], \quad (14)$$

where $y(t)$ is the Hilbert transform of the original process $x(t)$:

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau. \quad (15)$$

The integral in this formula is understood in the sense of the Cauchy principal value. For the stochastic process $x(t)$, the convergence of this integral is defined in the mean-square sense. Equation (14) implies the following relations:

$$x(t) = a(t)\cos\Phi(t), \quad y(t) = a(t)\sin\Phi(t). \quad (16)$$

Thus, the instantaneous amplitude $a(t)$ and phase $\Phi(t)$ of the process $x(t)$ are defined by

$$\Phi(t) = \arctan(y/x)+\pi k, \quad k = 0, 1, 2, 3, \ldots, \qquad a(t) = \sqrt{x^2(t)+y^2(t)}. \quad (17)$$

The time derivative of the instantaneous phase yields the instantaneous frequency

$$\omega(t) = \frac{1}{a^2(t)}[x(t)\dot{y}(t)-y(t)\dot{x}(t)]. \quad (18)$$

The mean frequency $\langle\omega\rangle$ is defined as

$$\langle\omega\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\omega(t)dt = \lim_{T\to\infty}\frac{1}{T}(\Phi(t_0+T)-\Phi(t_0)). \quad (19)$$

Finally, we can consider the instantaneous phase difference between input and output signals: $\varphi(t) = \Phi(t) - \omega_1 t$.

In certain special situations, one uses another definition of the instantaneous phase, which is based on the calculation of the return time of the phase trajectory to a Poincaré secant plane. Let us transform a continuous oscillatory process $x(t)$ into a discrete one, $x(t_k)$, where t_k are the time instants at which the phase trajectory successively intersects the Poincaré secant plane in one direction. The time interval between two successive intersection is $T(t) = t_{k+1} - t_k$, $t_k < t < t_{k+1}$. In this case, the instantaneous frequency of oscillations is defined by

$$\Phi(t) = 2\pi\frac{t-t_k}{t_{k+1}-t_k}+2\pi k, \quad t_k < t < t_{k+1}. \quad (20)$$

Thus, in this approximation, the phase $\Phi(t)$ represents a piecewise linear function of time. It is obvious that, for such a definition of phase, the mean frequency defined by (19) is expressed as

$$\langle\omega\rangle = \lim_{T\to\infty} 2\pi\frac{M(T)}{T}, \quad (21)$$

where $M(T)$ is the number of returns of the trajectory to the secant surface in time T.

Sometimes, one uses the third definition of the instantaneous phase, which can be considered as the generalization of the classical definition of phase to the case when the phase trajectory nonuniformly rotates around an unstable equilibrium point, so that the angular velocity is time-dependent. In this case, one can define the instantaneous phase as the rotation angle of the projection of a radius vector that specifies the position of a phase point on a certain plane of dynamic variables (x, y). Usually, the origin of the radius vector is located at the equilibrium point. One can introduce the change of variables

$$x(t) = a(t)\cos\Phi(t), \quad y(t) = a(t)\sin\Phi(t). \quad (22)$$

The instantaneous phase $\Phi(t)$ and amplitude $a(t)$ are defined by formulas similar to (17); however, instead of a Hilbert-adjoint process, one uses a dynamical variable $y(t)$. The instantaneous frequency ω of oscillations is introduced as the derivative of the instantaneous phase, while the mean frequency $\langle\omega\rangle$ is defined by (19).

2. SYNCHRONIZATION OF CHAOTIC OSCILLATIONS

The next step toward generalizing the principles of synchronization of periodic oscillations is the study of the synchronization phenomena of deterministic chaotic systems. The complex, nonperiodic, character of oscillations has required a new, more general, definition of synchronization. In [10, 32, 60], the authors suggested to define the synchronization as the establishment of certain relations between appropriately chosen functionals of oscillatory processes in partial systems. Such a definition is utterly generalized and can hardly be applied to solving concrete problems since, because of a different choice of the functionals of chaotic oscillations, the synchronization of chaos can be interpreted in several different senses. There exist the concepts of frequency–phase, full, and generalized synchronization of chaotic oscillations. The concept of frequency–phase synchronization of chaos seems to be the most consistent one, which represents a direct development of the classical theory of synchronization of periodic self-oscillations.

Frequency–Phase Synchronization of Chaos

The classical theory of synchronization is easily generalized to the case of a spiral-type chaotic regime [111, 112]. The spectrum of spiral chaos has a distinct peak at a frequency close to the frequency of the limit cycle that generates the chaotic attractor as a result of a sequence of subharmonic bifurcations. One can consider this frequency as a *base* frequency of oscillations and study the phenomena of locking or suppressing the base frequency. One can also introduce an instantaneous phase of chaotic oscillations and draw a certain analogy with the phase-locking phenomenon. In these cases, one speaks, respectively, of the frequency [49–53] or phase [54–56] synchronization of chaos. However, since the frequency and phase relations are closely connected, it would be convenient to apply the term of *frequency–phase synchronization* or the term of *Huygens synchronization* introduced in [32]. The frequency–phase synchronization of chaos can be observed in systems that sufficiently strongly differ both from the viewpoint of their mathematical models and by their dynamics. Also, there may exist a forced frequency–phase synchronization, including the one forced by a periodic signal.

The first attempts to generalize the classical ideas of synchronization as the locking or suppression of frequencies to the case of interaction between chaotic self-oscillators were made in [49–53, 113, 114]. It was found out that one can distinguish the domains of synchronization of chaos, similar to “Arnold tongues,” on the plane of parameters governing the coupling degree and the mismatch between the chaotic oscillators. The chaotic oscillations in these domains (synchronous chaos) are topologically different from the chaos beyond this domain (nonsynchronous chaos).

In [54–56, 115], the definitions of the instantaneous phase of chaotic oscillations were given, which were described in Section 1, and it was demonstrated that the phase locking occurs in interacting chaotic systems or under an external harmonic signal applied to a chaotic self-oscillator. The generalized definition of phase locking that can be applied to the case of chaotic oscillations is based on the requirement that the time variations in the instantaneous phase difference should be bounded [54]:

$$\lim_{t\to\infty}|\Phi_1(t) - \Phi_2(t)| < \text{const}. \quad (23)$$

Numerical experiments show that, although the current values of the instantaneous phases and frequencies determined by various methods may be somewhat different, the behavior of the mean frequencies proves to be virtually identical. Thus, any of the definitions of the instantaneous phase given above can be used for diagnosing the frequency–phase synchronization.

Obviously, the phenomena of frequency and phase locking in the case of chaotic oscillations, just as in the case of periodic oscillations, are closely interrelated.

As an example of frequency–phase synchronization of chaotic oscillations, let us consider the Anishchenko–Astakhov chaos generator under an external harmonic signal [51, 112]:

$$\begin{aligned} \dot{x} &= mx + y - xz + B\sin(\omega_1 t), \\ \dot{y} &= -x, \\ \dot{z} &= -gz + g(x + |x|)x/2. \end{aligned} \quad (24)$$

When $m = 1.1$ and $g = 0.3$, the regime of dynamical chaos is realized in the autonomous system (24); this regime is typical of systems with a saddle–focus separatrix loop (the Shil’nikov theorem [116, 112]). This chaos is characterized by a continuous spectrum against the background of which there exist spikes at the base frequency ω_2 and at its harmonics $n\omega_2$ and subharmonics $\omega_2/2n$. The value of the dimensionless base frequency ω_2 of the autonomous system (24) is close to unity. Let us choose the driving frequency $\omega_1 = 1 - \Delta \approx \omega_2 - \Delta$, where Δ is a mismatch, and consider the characteristics of oscillations for various values of the amplitude B of the external signal and the mismatch Δ.

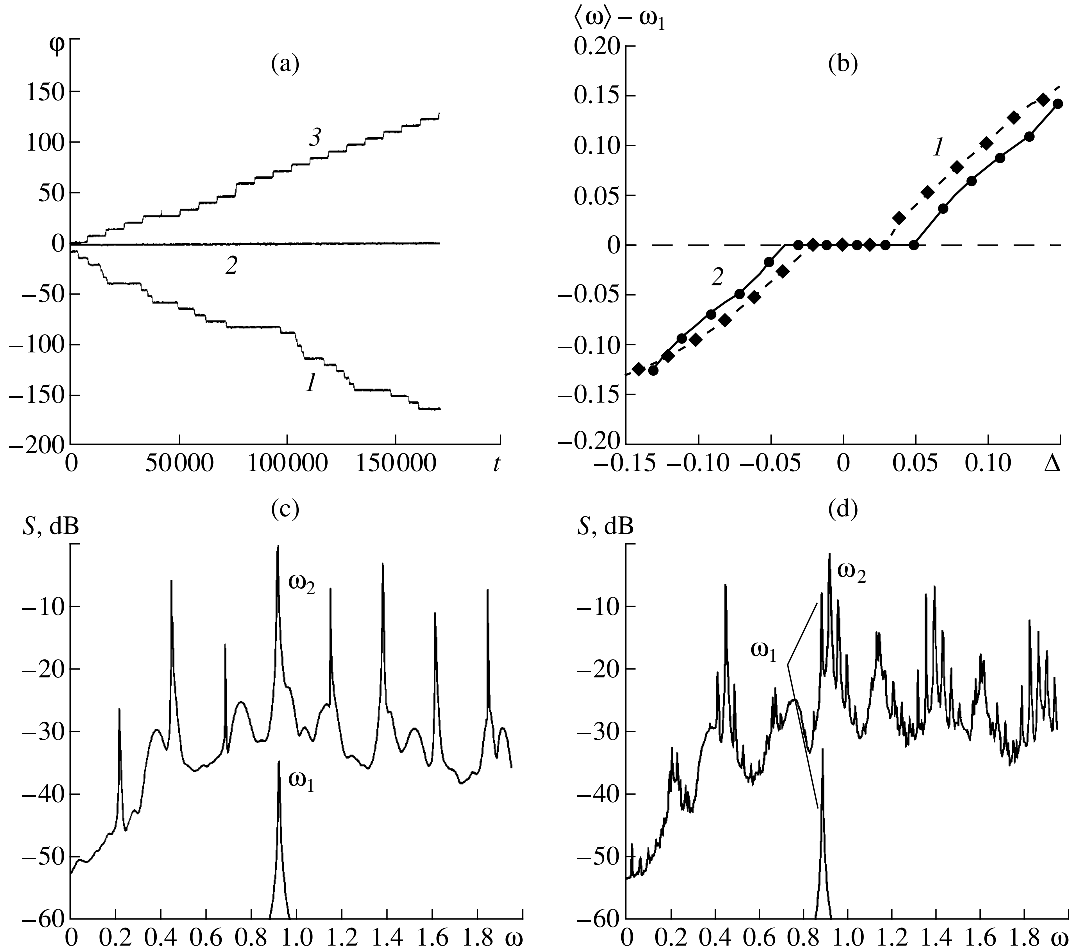

Fig. 6. The locking of the phase of chaotic oscillations in system (24) for $m = 1.1$ and $g = 0.3$; (a) evolution of the instantaneous phase difference for an external-signal amplitude of $B = 0.02$ and various values of the mismatch parameter Δ: (*1*) $\Delta = -0.02121$, (*2*) $\Delta = -0.02421$, and (*3*) $\Delta = 0.03171$; (b) the difference between the mean frequency of chaotic oscillations and the driving frequency as a function of the mismatch parameter for various amplitudes of the external signal: (*1*) $B = 0.02$ and (*2*) $B = 0.04$; (c and d) the power spectra of chaotic oscillations and external signal inside and outside the synchronization domain, respectively, for $B = 0.02$.

Let us define the instantaneous phase by (17) and calculate the time dependence of the phase difference $\varphi(t) = \Phi(t) - \omega_1 t$ for various values of the mismatch parameter. The results are represented in Fig. 6a.

The synchronization regime occurs for $\Delta = -0.02121$ and is destroyed as the mismatch increases. Using the definition of mean frequency (19), we calculate the difference $\langle\omega\rangle - \omega_1$ as a function of mismatch. Figure 6b shows that, in a finite interval of the mismatch parameter, the mean frequency of chaotic oscillations is locked by the external signal. Here, the width of the synchronization domain depends on the amplitude B of the external signal. Thus, we observe the locking of the base frequency of chaotic oscillations. Physically, this effect is demonstrated in Figs. 6c and 6d, which represent the power spectra of oscillations of the oscillator and the spectrum of the external signal in the synchronization domain for $\Delta = 0.01$.

One can see that the base frequency ω_2, which differs from the driving frequency ω_1 by Δ, is tuned to the driving frequency as a result of synchronization. The locking of the base frequency is accompanied by the locking of the mean frequency and the instantaneous phase of chaotic oscillations, which were defined by the concept of analytic signal on an unbounded time interval. The fragment of the synchronization domain of system (24) on the plane of parameters is shown in Fig. 7.

As we have already mentioned, the specific feature of system (24) is the presence of a pronounced base frequency, which virtually coincides with the mean frequency of oscillations ($\omega_2 = \langle\omega\rangle$). A similar scenario is observed in all the other cases where a spiral attractor of Shil'nikov exists in the phase space of the chaotic system to be synchronized (for example, the Rössler system, Chua's circuit, the Lorentz systems with large

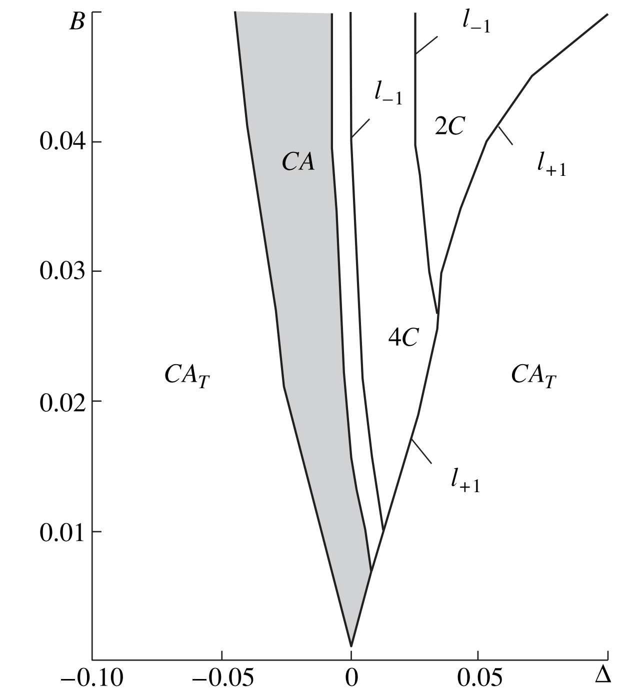

Fig. 7. Synchronization domain of system (24) on the plane of parameters "amplitude–mismatch" for $m = 1.1$ and $g = 0.3$. The hatched domain CA corresponds to synchronous chaos. The domains $2C$ and $4C$ exhibit periodic synchronous oscillations with periods T_1 and $2T_1$, respectively ($T_1 = 2\pi/\omega_1$), CA_T is a domain of nonsynchronous chaos. The lines l_{-1} correspond to period-doubling bifurcations of cycles, while the lines l_{+1} correspond to tangent bifurcations.

values of the Rayleigh parameter, and the Belousov–Zhabotinskii system [52, 54, 115]).

Saddle cycles embedded into a synchronous attractor play an important role in the destruction and onset of frequency–phase synchronization. In [51], it was established that the lines of tangent bifurcations of saddle cycles are concentrated on the plane of control parameters at the synchronization boundary of chaos.[1] The boundary of chaotic synchronization is interpreted as a critical line to which the points of tangent bifurcations tend as the periods of cycles increase. The role of the saddle cycles embedded into a chaotic attractor were considered in detail in [55] by using an irreversible two-dimensional map simulating the synchronization of the chaotic oscillator by an external periodic force. This map is given by

$$\begin{aligned} x_{n+1} &= f(x_n, \varphi_n), \\ \varphi_{n+1} &= \varphi_n + \Omega + \varepsilon\cos(2\pi\varphi_n + g(x_n)), \quad \mathrm{mod}1, \end{aligned} \tag{25}$$

where φ is the phase difference of the oscillator and the external force, Ω is a frequency mismatch, and ε characterizes the amplitude of the external force. The function $g(x)$, which defines the chaotic modulation of the phase, is chosen as $g(x) = \delta x$, while the function $f(x, \varphi)$, which describes the dynamics of the chaotic oscillator amplitude, was chosen as $f(x, \varphi) = 1 - a|x| + \varepsilon\rho\sin(2\pi\varphi)$.

A similar map was also considered in [56]. The saddle cycles that constitute a "skeleton" of synchronous chaos undergo tangent bifurcations together with appropriate periodic repellers. The latter constitute the skeleton of a chaotic repeller that is tangent to the chaotic attractor at certain points (namely, at the points of saddle and unstable cycles at the moment when they merge together). Each pair of skeleton cycles belongs to an unstable invariant curve (that corresponds to a saddle torus of a flow system). As a result of tangent bifurcation, the motion along an invariant curve becomes ergodic; i.e., a direction appears such that the phase point moves away from the synchronous attractor and comes back making a turn along the invariant curve. In this case, one observes a phase difference of 2π.

This mechanism of breaking the phase synchronism of chaos was confirmed by an example of the Rössler oscillator [117]. The vanishing of individual pairs and the rise of instability directions may not be exhibited for a long time in a numerical experiment. However, the accumulation of local variations in the structure of synchronous chaos ultimately leads to the final destruction of the phase synchronization. The corresponding bifurcation is well diagnosed by the behavior of Lyapunov exponents. On the boundary of the phase locking, one of the negative Lyapunov exponents vanishes [54, 115].

Under a considerable frequency mismatch and a large amplitude of signal (or a large coupling coefficient of two chaotic oscillators), one can observe the suppression of chaotic self-oscillations [50–53]. In this case, there exist periodic regimes in the domain of synchronization, while the boundary of this domain corresponds to the bifurcation of birth of a torus from a limit cycle (just as in the classical case of suppression of periodic self-oscillations).

Full Synchronization of Chaos

Consider a system of interacting oscillators defined by identical equations

$$\begin{aligned} \dot{\vec{x}}_1 &= \vec{F}(\vec{x}_1, \vec{\alpha}_1) + \gamma_1\vec{g}(\vec{x}_1, \vec{x}_2), \\ \dot{\vec{x}}_2 &= \vec{F}(\vec{x}_2, \vec{\alpha}_2) + \gamma_2\vec{g}(\vec{x}_2, \vec{x}_1), \end{aligned} \tag{26}$$

where $\vec{x}_{1,2} \in R^N$, $\vec{\alpha}_{1,2}$ are vector parameters of the partial systems, $\gamma_{1,2}$ are coupling parameters, and the function $\vec{g}$ determines the character of coupling; if $\vec{x}_1 = \vec{x}_2$, then $\vec{g}(\vec{x}_1, \vec{x}_2) = 0$. If $\vec{\alpha}_1 = \vec{\alpha}_2 = \vec{\alpha}$, then the partial oscillators are completely identical. In this case, for certain values of the coupling parameter $\gamma_{1,2}$, partial

[1] A tangent bifurcation is a saddle–node bifurcation when the leading multiplicator of the cycle is equal to +1 [112].

oscillations may become completely identical ($\vec{x}_1(t) \equiv \vec{x}_2(t)$), while remaining chaotic. In a number of works [42–47], this phenomenon was referred to as a chaotic synchronization. In contrast to the case of frequency–phase synchronization considered above, it is more appropriate to call it a *full (in-phase)* synchronization. In systems of the type (26), the full synchronization can be observed in the case of symmetric ($\gamma_1 = \gamma_2 = \gamma$) as well as nonsymmetric ($\gamma_1 \neq \gamma_2$) and unidirectional ($\gamma_1 = 0$) coupling of partial systems.

When $\vec{\alpha}_1 = \vec{\alpha}_2 = \vec{\alpha}$, there exists an invariant manifold U, $\vec{x}_1 = \vec{x}_2$, called a *symmetric subspace*, in the phase space of system (26). The phase trajectories lying in U correspond to fully synchronous oscillations. If the limit set belonging to U attracts the phase trajectories not only from U, but also from a certain neighborhood of the symmetric subspace, then one can observe a full synchronization. The full synchronization is possible not only when the system interacts with spiral chaos, but also in the case of chaotic dynamics of other types (for example, in the case of the Lorentz attractor [81] and Chua's double scroll [46]).

In addition, the full synchronization can be observed when a slave system is affected by a master system [47]. The master system is represented as a combination of two parts (subsystems),

$$\dot{\vec{v}} = \vec{p}(\vec{v}, \vec{y}_1), \quad \dot{\vec{y}}_1 = \vec{q}(\vec{v}, \vec{y}_1), \tag{27}$$

while the slave system is chosen to be identical to one of these parts. Then, the two interacting systems (the master and slave systems) are described by the equations

$$\dot{\vec{v}} = \vec{p}(\vec{v}, \vec{y}_1), \;\; \dot{\vec{y}}_1 = \vec{q}(\vec{v}, \vec{y}_1), \;\; \dot{\vec{y}}_2 = \vec{q}(\vec{v}, \vec{y}_2), \tag{28}$$

where $\vec{v} \in R^m$ and $\vec{y}_{1,2} \in R^{N-m}$. In the full synchronization regime, $\vec{y}_1(t) \equiv \vec{y}_2(t)$.[2]

The robustness of the full chaotic synchronization regime and the mechanisms of its destruction were considered in [123–135]. In these works, it is investigated how an invariant manifold U of system (26) that contains an in-phase chaotic attractor ceases to be an attracting manifold. The stability of the trajectories of (26) that belong to the invariant manifold U with respect to a small transverse perturbation $\vec{u} = \vec{x}_2 - \vec{x}_1$ is determined by the conditional Lyapunov exponents [46]

$$\lambda_c^j = \lim_{t \to \infty} \frac{1}{t} \ln \left\| \vec{u}^j(t, \vec{x}_1(0)) \right\|, \quad j = 0, 1, 2, 3, \ldots, N, \tag{29}$$

where $\vec{u}^j$ is the jth fundamental solution of the linearized system

$$\dot{\vec{u}} = \frac{\partial}{\partial \vec{x}_2} [\vec{F}(\vec{x}_2, \alpha) + \gamma_2 \vec{g}(\vec{x}_1, \vec{x}_2) - \gamma_1 \vec{g}(\vec{x}_2, \vec{x}_1)] \Big|_{\vec{x}_2 = \vec{x}_1} \cdot \vec{u}. \tag{30}$$

If all λ_y^j, $j = 1, 2, \ldots, N$, are negative, then the regime of full chaotic synchronization is asymptotically stable. When at least one λ_c^j becomes positive, the invariant manifold ceases to be stable. As a result, the regime of full synchronization of chaos, called a blowout bifurcation [127], is destroyed. This destruction is usually accompanied by transient (on finite time intervals) or "true" intermittency (the Yamada–Fujisaki, or "on–off," intermittency [124–127, 136]).

However, conditional Lyapunov exponents represent the characteristics averaged over the attractor that do not diagnose all local variations in the structure of the limit set. It was shown in [123–127] that, even before the onset of instability in the transverse direction, a countable set of points of the invariant manifold may arise that are characterized by transversal instability. These points belong to unstable cycles lying in the invariant subspace. Having gotten into the neighborhood of such a cycle, the phase point (provided that it does not lie strictly in U) moves away from the invariant manifold. If there are no other attractors than the in-phase one (i.e., the one lying in U) for these values of the parameters, then, after a while, the trajectory again returns to this neighborhood and then falls into the invariant manifold. In this case, one can observe a long transient process of on–off intermittency. The effect of small noise on the in-phase chaos leads to a continual resumption of the intermittency process. As a result of the influence of noise, the experimentally observed chaotic attractor does not belong to the invariant manifold any longer; it as if swells up. This phenomenon is called *bubbling* [124–127].

The presence of unstable cycles in the in-phase chaos leads to the formation of domains whose cross sections remind one of beaks with the sharp ends resting on the points of unstable cycles [124, 125, 131]. A trajectory starting from such a domain moves away from the in-phase attractor. If there exists a certain out-of-phase attractor in the system for the same values of parameters, then the phase trajectory gets into this attractor. The appearance of a countable set of such beaks in the phase space results in the "riddling" of the local neighborhood of the chaotic attractor lying in U. This phenomenon was called *riddling* [124–129, 134]. As a result, the chaotic attractor in U is not an attractor in the conventional sense any longer; it is now called a *Milnor attractor* [137]. Figure 8 demonstrates the riddled neighborhood of a chaotic attractor lying in U (on the bisector) for a system of coupled logistic maps. For

[2] An interest in the problem of full synchronization was stimulated by the fact that it can be used in the development of secure communication systems [118–122].

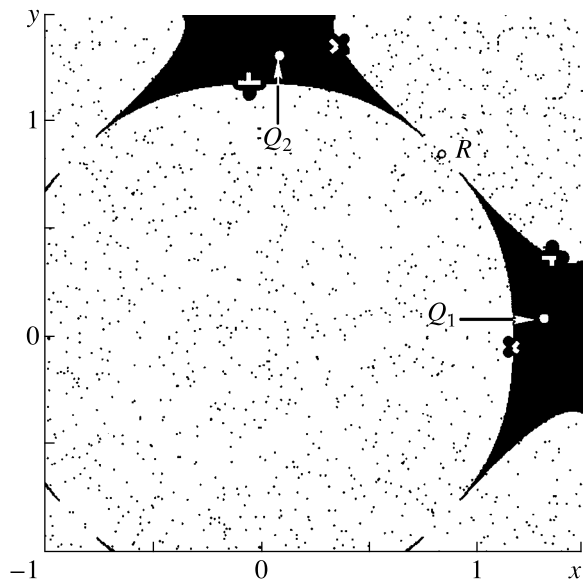

Fig. 8. Bistability and riddling in a system of coupled logistic maps (41) for $r = 1.56$ and $\gamma = 0.143$. In addition to a chaotic in-phase attractor situated on the bisector $x = y$, there exists an out-of-phase stable cycle of period 2 (it contains the points $Q_{1,2}$). The attraction basin of the out-of-phase attractor is marked by dark dots.

the values of the parameters considered, the transversal Lyapunov exponent is still negative; i.e., a blowout bifurcation has not yet occurred. When the transverse direction becomes unstable in the mean on the attractor, the chaotic limit set lying in the invariant manifold ceases to be an attractor even in the sense of Milnor, which corresponds to the blowout bifurcation.

Delay Synchronization

The introduction of a parameter mismatch between partial systems leads to the vanishing of the symmetric subspace; hence, a full synchronization is impossible for a finite value of γ. If the parameter mismatch does not lead to a change in the structure of the chaotic attractor, but only changes the base frequency of chaotic oscillations, then, starting from a certain coupling level, one can observe the so-called *lag synchronization* [59]. The oscillations of the oscillators completely repeat each other with a certain time delay τ_d: $\vec{x}_1(t) \equiv \vec{x}_2(t + \tau_d)$. Although there does not exist an invariant subspace in the case of lag synchronization, the chaotic attractor is topologically equivalent to the in-phase attractor in the case of full synchronization. Therefore, we can interpret the lag synchronization as a generalization of the concept of full synchronization to the case of systems with a weak parameter mismatch.

Thus, we can distinguish three types of chaos synchronization sin the case of interacting chaotic oscillators: frequency–phase synchronization, lag synchronization, and full synchronization. The boundary of the synchronization domain is determined by the Huygens synchronization, i.e., corresponds to the locking of the instantaneous phases and base frequencies of chaotic oscillations of partial systems or to the suppression of self-oscillations in one of the oscillators. In the latter case, despite the chaotic behavior of both partial systems in the autonomous regime, periodic oscillations occur in the synchronization domain. A decrease in the mismatch and an increase in the coupling parameter may give rise to a stronger synchronization, the lag synchronization. The transition from the frequency–phase synchronization to the lag synchronization is a complex process; the bifurcation mechanism of this process has not yet been sufficiently studied. For zero mismatch, starting from a certain value of the coupling parameter, one can observe a full synchronization: chaotic trajectories in a symmetric subspace become stable in the entire phase space of the system.

Generalized Synchronization

The concept of generalized synchronization of chaotic systems was introduced in [48, 57, 58]. Two systems are considered to be synchronized if there is a functional (static) relation between the states of these systems. In [58], the authors proposed the theory of generalized synchronization for chaotic systems with unidirectional coupling,

$$\dot{\vec{x}} = \vec{F}(\vec{x}), \quad \dot{\vec{y}} = \vec{G}(\vec{y}, \vec{V}) = \vec{G}(\vec{y}, \vec{h}(\vec{x})), \qquad (31)$$

where $\vec{x} \in \mathrm{R}^N$ and $\vec{y} \in R^n$ are the state vectors of the first and second systems, respectively. The vector $\vec{V}$ is determined by the instantaneous state of the first system, $\vec{V} = \vec{h}(\vec{x}) \in R^k$. The first and second partial systems (31) are called master and slave systems, respectively. In the case of generalized synchronization, the relation $\vec{y} = \vec{Q}(\vec{x})$ is established between $\vec{x}$ and $\vec{y}$, and all phase trajectories from a certain attraction domain B tend to the manifold $M = \{\vec{x}, \vec{Q}(\vec{x})\}$. If $\vec{Q}$ is the identity transformation, then the generalized synchronization degenerates into a full synchronization. It was shown that the generalized synchronization in (31) occurs if and only if the slave system $\dot{\vec{y}} = \vec{G}(\vec{y}, \vec{h}(\vec{x}))$ is asymptotically stable; i.e., for any initial states $\vec{y}_1(0)$ and $\vec{y}_2(0)$ from a certain domain $B_{\vec{y}}$, there exists the limit

$$\lim_{t \to \infty} \left\| \vec{y}(t, \vec{x}(0), \vec{y}_1(0)) - \vec{y}(t, \vec{x}(0), \vec{y}_2(0)) \right\| = 0. \quad (32)$$

When this condition is fulfilled, the generalized synchronization is observed even in the case when the slave and master systems are completely different. The presence of synchronization is easily diagnosed by conditional Lyapunov exponents

$$\lambda_c^j = \lim_{t\to\infty}\frac{1}{t}\ln\left\|\vec{z}^j(t, \vec{x}(0), \vec{y}(0))\right\|, \quad j = 1, 2, \ldots, n, \quad (33)$$

where $\vec{z}^j$ is the jth fundamental solution of the linearized equation

$$\dot{\vec{z}} = \left[\frac{\partial \vec{G}(\vec{y}, \vec{h}(\vec{x}))}{\partial \vec{y}}\right]\Bigg|_{\vec{y} = \vec{Q}(\vec{x})} \cdot \vec{z}. \quad (34)$$

In the synchronization regime, all conditional exponents λ_c^j of the slave system should be negative.

Various manifestations of the chaotic synchronization are observed not only in numerical simulation, but also in many full-scale experiments carried out mainly on radio-communication systems. The locking and suppression of the base frequencies of chaotic oscillations were experimentally investigated in [50, 51]. The locking of the phase of chaotic self-oscillations was considered in [138]. The full and almost full synchronizations of chaos were observed in the experiments described in [45, 48, 139, 140]. The blowout bifurcation and the associated bubbling and riddling phenomena were experimentally investigated in [125], while the lag synchronization was studied in [141]. In the experiments with two unidirectionally coupled lasers [142], the authors observed generalized, phase, and lag synchronizations of chaotic oscillation regimes. Various manifestations of synchronization were interpreted as certain stages of nonlinear interaction between systems.

Characteristics of the Degree of Synchronism between Chaotic Oscillators

Various qualitative characteristics of the synchronism degree between partial oscillators are used for two nearly identical interacting oscillators. In [55], the minimal value of the similarity function

$$\kappa = \min_{\tau} S(\tau), \quad (35)$$

where $S(\tau)$ is the similarity function defined by

$$S^2(\tau) = \frac{\langle (x_2(t+\tau) - x_1(t))^2\rangle}{\sqrt{\langle x_1^2(t)\rangle\langle x_2^2(t)\rangle}}. \quad (36)$$

was proposed as the degree of synchronism between the oscillatory processes $x_1(t)$ and $x_2(t)$ of partial systems. In the case of full synchronization and lag synchronization, $\kappa = 0$. As the mismatch increases and the coupling decreases, κ increases. The normalized mutual correlation function

$$R_{x_1x_2}(\tau) = \frac{\langle x_1(t)x_2(t+\tau)\rangle - \langle x_1(t)\rangle\langle x_2(t+\tau)\rangle}{\sqrt{(\langle x_1^2(t)\rangle - \langle x_1(t)\rangle^2)(\langle x_2^2(t+\tau)\rangle - \langle x_2(t+\tau)\rangle^2)}} \quad (37)$$

serves as another characteristic of the synchronism degree between chaotic oscillators. The quantity $\eta = \max_{\tau} R_{x_1x_2}(\tau)$ equals unity in the case of lag synchronization (and, of course, in the case of full synchronization) and tends to zero when the statistical link between $x_1(t)$ and $x_2(t)$ is lost. The density of probability of the instantaneous phase difference $\varphi(t) = \Phi_1(t) - \Phi_2(t)$ of chaotic oscillations, defined on the interval $[-\pi, \pi]$, also characterizes the degree of the synchronism of motion. One can choose the effective diffusion coefficient D_{eff} of the instantaneous phase difference, the variance of the instantaneous phase difference defined on the interval $[-\pi, \pi]$, or the entropy of the variance distribution (corresponding to a fixed discretization step of φ) as the quantitative characteristics.

The synchronism degree can also be evaluated within the spectral approach. To this end, one employs the coherence function [51]

$$r_{x_1x_2}(\omega) = \frac{|W_{x_1x_2}(\omega)|}{\sqrt{W_{x_1}(\omega)W_{x_2}(\omega)}}, \quad (38)$$

where W_{x_1} and W_{x_2} are the power spectra of the centered processes $x_1(t) - \overline{x_1}$ and $x_2(t) - \overline{x_2}$ and $W_{x_1x_2}$ is their mutual spectrum. For statistically independent processes, $r_{x_1x_2} \equiv 0$, whereas, for linear dependence between $x_1(t)$ and $x_2(t)$, we have $r_{x_1x_2} \equiv 1$. To introduce a frequency-independent quantitative characteristic, one can take the mean value of the coherence ratio in the frequency interval considered, $\bar{r} = \int_{\omega_1}^{\omega_2} r_{x_1x_2}(\omega)d\omega$.

A characteristic of the degree of consistency between the phases of various spectral components of signals $x_{1,2}(t)$ was proposed in [51]. The phase spectra $\varphi_{1,2}(\omega) = \arg[\int_{-T/2}^{T/2} x_{1,2}(t)\exp(-j\omega t)dt]$ of the oscillations $x_1(t)$ and $x_2(t)$ were considered on a finite time interval T. The current phase difference

$$\Delta\varphi(\omega) = \varphi_1(\omega) - \varphi_2(\omega), \quad \Delta\varphi \in [-2\pi, 2\pi] \quad (39)$$

was introduced for any frequency of the spectrum. For different initial points on the attractor, one obtains different functions $\Delta\varphi(\omega)$; thus, one can calculate the probability density $p(\Delta\varphi, \omega)$ on the set of such functions. In the case of full and lag synchronizations, the probability density is given by a frequency-independent δ-function. Under the failure of lag synchronization,

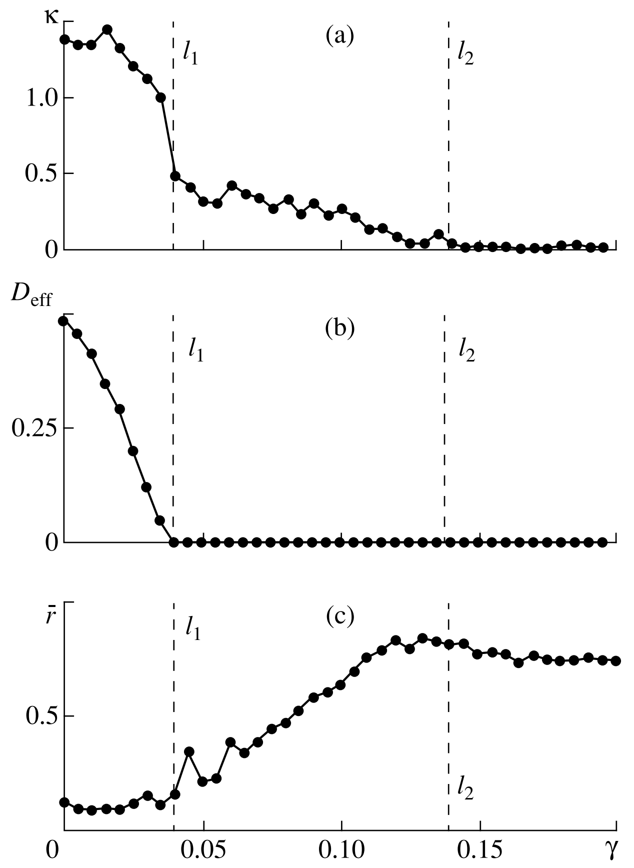

Fig. 9. Comparison of the behavior of different quantitative characteristics of the synchronism degree between interacting oscillators depending on the coupling parameter for model (40) with $\alpha = 0.165$, $\beta = 0.2$, $\mu = 10$, $\Omega_0 = 0.97$, and $\Delta = 0.02$; (a) the minimal value of the similarity function $\kappa(\gamma)$, (b) the effective diffusion coefficient $D_{eff}(\gamma)$ of the instantaneous phase difference, and (c) the mean value of the coherence ratio $\bar{r}(\gamma)$. The dashed lines represent the boundaries of the domains of phase synchronization (l_1) and lag synchronization (l_2).

the distribution has finite width and shape that are different at different frequencies. The variance or the entropy of the phase-difference distribution averaged over all frequencies may serve as a quantitative characteristic of the degree of synchronism between oscillators.

As an example, Fig. 9 represents certain characteristics of the synchronism degree between partial chaotic oscillators versus the coupling parameter for a fixed frequency mismatch that were calculated for a system of two coupled Rössler oscillators given by

$$\begin{aligned}
\dot{x}_1 &= -\omega_1 y_1 - z_1 + \gamma(x_2 - x_1),\\
\dot{x}_2 &= -\omega_2 y_2 - z_2 + \gamma(x_1 - x_2),\\
\dot{y}_1 &= \omega_1 x_2 + \alpha y_1, \quad \dot{y}_2 = \omega_2 x_2 + \alpha y_2,\\
\dot{z}_1 &= \beta + z_1(x_1 - \mu), \quad \dot{z}_2 = \beta + z_2(x_2 - \mu),
\end{aligned} \tag{40}$$

where $\omega_{1,2} = \Omega_0 \pm \Delta$ are the parameters determining the frequencies of partial oscillators, Δ is the mismatch, γ is the coupling parameter, and α and μ control the dynamical regime of oscillators. The graph of the minimal value of the similarity function κ is shown in Fig. 9a. Figure 9b represents the diffusion coefficient D_{eff}, and Fig. 9c, the graph of the mean value of the coherence ratio $\bar{r}$.

The effective diffusion coefficient D_{eff} allows one to reliably diagnose the boundary of the domain of frequency–phase synchronization of chaos (the dashed line l_1 in Fig. 9). The effective diffusion coefficient vanishes everywhere within the domain of synchronization; therefore, one cannot determine the transition to the lag synchronization or full synchronization using this coefficient. The boundary of the domain of lag synchronization (the dashed line l_2 in Fig. 9) can be determined from the vanishing of the minimal value of the similarity function, which is clearly illustrated in Fig. 9a. The averaged coherence ratio $\bar{r}$ demonstrates the sensitivity both to the phase locking and to the onset of lag synchronization. However, this sensitivity manifests itself in the character of the dependence on the coupling parameter, rather than in the absolute value of r. As the coupling parameter increases, the coherence ratio increases starting from the locking boundary up to the boundary of lag synchronization.

Synchronization and Multistability

A characteristic feature of the mutual synchronization of oscillators with period-doubling bifurcations is the presence of multistability in the domain of synchronization [42, 51, 53, 83, 139, 143–145]. This multistability can be called phase multistability since it is related to the possibility of mutual synchronization of oscillations whose spectrum includes the oscillations at the subharmonics of the base frequency ω_0 with various phases with respect to each other. The number of possible synchronous regimes that differ by a phase shift between partial oscillators is the greater the more subharmonics of the base frequency can be distinguished in the spectrum.

For the initial oscillations with period T_0, the phase difference φ_0 between partial oscillators is analogous to the phase difference $\varphi_0 \pm 2\pi m$, $m = 1, 2, \ldots$. For period-doubled ($2T_0$) oscillations whose spectrum contains the subharmonic $\omega_0/2$, the phase differences φ_0 and $\varphi_0 \pm 2\pi$ correspond to two different limit cycles. The number of possible limit cycles with period $2^k T_0$ increases up to 2^k. These cycles differ by a phase shift between partial oscillators that may take values of $\varphi_0 + 2\pi m$, where $m = 0, 2, \ldots, 2^{k-1}$. The phase multistability that has appeared in the domain of periodic oscillations is preserved in the chaos domain if the chaotic attractor has a shape of a band with 2^k turns around a saddle–focus. The hierarchy of phase multistability in identical systems with dissipative coupling was thoroughly investigated during the numerical simulation of the dynamics of coupled logistic maps [139, 143] and in the experiments with in-phase excited nonlinear radio contours [144]. The revealed hierarchy of regimes possesses certain features of universality that also manifest themselves in a dissipative interaction between self-oscillating flow systems [51, 53, 145].

The basic model for investigating the phase multistability in identical dissipatively coupled oscillators is represented by a system of coupled logistic maps of the form [42, 143, 146]

$$\begin{aligned} x_{n+1} &= r - x_n^2 + \gamma(x_n^2 - y_n^2), \\ y_{n+1} &= r - y_n^2 + \gamma(y_n^2 - x_n^2), \end{aligned} \tag{41}$$

where r is the control parameter of a logistic map and γ is the coupling parameter. For the discrete system (41), a phase shift is a shift between time-realizations in the subsystems on m iterations. The oscillations with $m = 0$ are called *in-phase* oscillations, and the limit sets corresponding to these oscillations belong to the symmetric subspace $x = y$. Oscillations with $m \neq 0$ are *out-of-phase* and do not belong to the symmetric subspace.

The development of the phase multistability as the parameter r increases occurs according to the following scenario. The initial cycle looses stability and becomes a saddle cycle. As r increases, the saddle cycle once again undergoes a period-doubling bifurcation. This bifurcation is simultaneously a symmetry-breaking bifurcation. The arising period-doubled cycle does not belong to the symmetric subspace any longer; however, it possesses reflection symmetry with respect to the line $x = y$. This cycle, born as a saddle cycle, becomes stable as r further increases. Each in-phase cycle generates its own branch of out-of-phase regimes. For the out-of-phase cycles that are originated from in-phase cycles, there certainly exists a bifurcation giving birth to a torus. As a result of resonance on a torus, new pairs of cycles arise, etc. The multistability is also preserved in the domain of chaotic regimes. Figure 10 represents the phase portraits of the limit cycles of system (41) corresponding to different m and the chaotic attractors generated by these cycles.

The transition to the analysis of coupled self-oscillating flow systems allows one to introduce a frequency mismatch between partial systems and analyze the domain of phase synchronization. The introduction of any parameter mismatch breaks the symmetry relations between the limit cycles, and the character of certain bifurcations of limit cycles is changed [133]. However, the analysis of system (40) and other similar systems has shown that a small frequency mismatch ($\Delta \leq 0.001$) under weak coupling ($\gamma \approx 0.02$) does not lead to a substantial variation in the evolution scheme of various types of oscillations as it occurs in the discrete model (41). A more significant mismatch between the frequencies of partial systems may lead to substantial

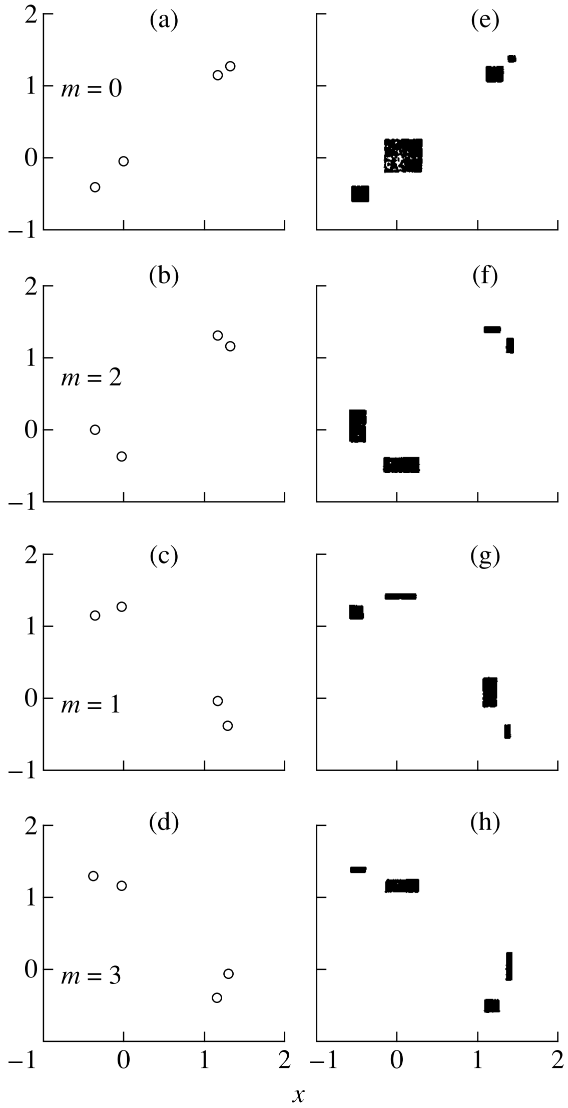

Fig. 10. Phase multistability in system (41) for $\gamma = 0.002$. (a–d) Cycles corresponding to various values of the phase shift m for $r = 1.3$ and (e–h) chaotic attractors generated by these cycles for $r = 1.415$.

variations in the order and character of bifurcations of periodic and chaotic regimes.

3. SYNCHRONIZATION OF STOCHASTIC OSCILLATIONS

Bistable Oscillator

There exist examples of systems in nature in which oscillations arise only under the influence of noise. In other words, a process $x(t)$ becomes oscillatory with a certain characteristic time only under the influence of noise. Generally, such systems can be referred to as self-oscillating systems since they generate an oscillatory process $x(t)$ whose statistical characteristics are independent of initial conditions within certain limits. A natural question arises concerning the possibility of

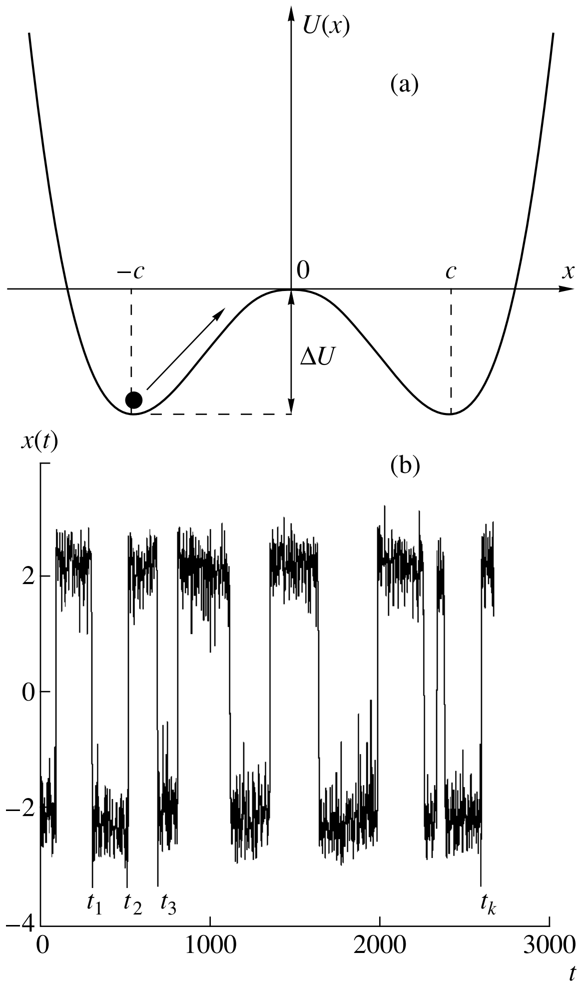

Fig. 11. (a) Bistable potential of system (43) and (b) realization $x(t)$ obtained by integrating system (44).

realizing a phase synchronization regime of stochastic oscillations.

As an example, let us consider the motion of a particle in the field of potential forces,

$$m\ddot{x} + \nu\dot{x} + \frac{dU(x)}{dx} = 0, \tag{42}$$

where m is the particle mass, ν is the dissipation coefficient, and $U(x)$ is the potential. When $U(x) = \frac{1}{2}\omega_0^2 x^2$, we obtain the equation of a linear oscillator. We will focus on the motion of a particle in the bistable potential

$$U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4.$$

Under the assumption of large friction ($\nu \gg 1$), we can neglect the inertial properties of the particle ($\ddot{x} \ll \dot{x}$) and reduce Eq. (42) to

$$\dot{x} = \alpha x - \beta x^3, \tag{43}$$

where $\alpha = am/\nu$ and $\beta = bm/\nu$ are parameters that determine the depth of the potential wells and the slope of their edges. System (43) has three equilibrium states: two stable nodes $x = \pm c$ and a saddle point at the origin of coordinates (see Fig. 11a). In the absence of an external signal, the particle relaxes toward one of the two stable states (a particular state is determined by the initial conditions) and remains there for an arbitrarily long period of time. Let us add a source of white noise of intensity D to (43):

$$\dot{x} = \alpha x - \beta x^3 + \sqrt{2D}\xi(t). \tag{44}$$

The stochastic system (44) acquires new physical properties as compared with (42): under the influence of noise, the particle not only performs small-scale fluctuations in the neighborhoods of equilibrium states but also randomly jumps from one potential well into another (see Figure 10b). These switchings are attributed to overcoming the potential barrier ΔU by the particle at random moments t_k when the noise energy exceeds an appropriate threshold level. System (44) is called a *stochastic bistable oscillator.* Thus, a new characteristic time T_e appears in system (44), the mean exit time from a metastable state. This characteristic time depends on the noise intensity D and can be considered as one of the time scales of stochastic oscillations. The theoretical study of the probability characteristics of system (41) was first carried out in the pioneering work of L.S. Pontryagin, A.A. Andronov, and A.A. Vitt [147]; in this work, they obtained a rigorous expression for the mean exit time from a potential well.

Noise-induced oscillations can be characterized by the mean frequency of switchings, $1/T_e$. In the case of the bistable oscillator (44) with a high potential barrier, this frequency is expressed by the Kramers approximate formula [148]

$$r_K = \frac{1}{2\pi}[|U''(0)|U''(c)]^{1/2}\exp\left(-\frac{\Delta U}{D}\right) \tag{45}$$

and is called the *Kramers frequency.*

Since system (44) can be considered as an autooscillating system in the sense considered above, it is natural to set up a problem of synchronization. Is it possible to synchronize the switching process in (44) by adding an additive harmonic force? If this is possible, then what principles will describe this phenomenon?

Phase Synchronization of Stochastic Switchings in a Bistable Oscillator

Consider a nonautonomous model of an overdamped bistable oscillator, which is a classical example for investigating the stochastic resonance and stochastic synchronization [94]:

$$\dot{x} = \alpha x - \beta x^3 + \sqrt{2D}\xi(t) + A\cos(\Omega_0 t + \psi_0). \tag{46}$$

System (46) does not have a deterministic natural frequency. At the same time, this system is characterized by a noise-controlled time scale—the mean exit time from a potential well, which corresponds to the mean switching frequency in the frequency domain. A periodic signal serves as an external "clock" for the bistable oscillator.

Suppose that α, $\beta > 0$, $\psi_0 = 0$, and the modulation amplitude A is small as compared to the potential barrier in SDE (46):

$$A < A_0 = \frac{2}{\sqrt{\beta}}\left(\frac{\alpha}{3}\right)^{3/2}. \tag{47}$$

Moreover, suppose that the modulation frequency is small as compared to the frequency of intrawell relaxation oscillations and that an adiabatic approximation is applicable. Using the definition of an analytic signal (14) with respect to the variable $x(t)$ satisfying Eq. (46), we obtain the following SDE for the analytic signal $w(t)$:

$$\dot{w} = \alpha w - \frac{\beta}{4}(3a^2 w + w^3) + \Xi(t) + A\exp(i\Omega_0 t), \tag{48}$$

where $\Xi(t) = \xi(t) + i\eta(t)$ is analytic noise and $\eta(t)$ is the Hilbert transform (15) of the original Gaussian noise $\xi(t)$. From Eq. (48), one can easily obtain the following equations for the instantaneous amplitude and phase of stochastic oscillations:

$$\begin{aligned} \dot{a} &= \alpha a - \frac{\beta}{2}a^3[1 + \cos^2(\varphi + \Omega_0 t)] + A\cos\varphi + \xi_1(t), \\ \dot{\varphi} &= -\Omega_0 - \frac{A}{a}\sin\varphi - \frac{\beta}{4}a^2\sin[2(\varphi + \Omega_0 t)] + \frac{1}{a}\xi_2(t), \end{aligned} \tag{49}$$

where $\varphi(t) = \Phi(t) - \Omega_0 t$ is the instantaneous phase difference, while the noise sources $\xi_{1,2}(t)$ are given by

$$\begin{aligned} \xi_1(t) &= \xi(t)\cos\Phi + \eta(t)\sin\Phi, \\ \xi_2(t) &= \eta(t)\cos\Phi - \xi(t)\sin\Phi. \end{aligned} \tag{50}$$

One can see that Eqs. (49) are very similar to the classical system of truncated equations (4). A substantial difference between these equations is that the second equation in (49) involves the frequency of an external signal instead of a mismatch. This fact stresses once again that the bistable system considered does not have a deterministic time scale that would correspond to a certain periodic motion.

To calculate the instantaneous phase, it is most reasonable to first integrate the initial SDE (49) using a certain known algorithm (in our calculations, we applied a method similar to the fourth-order Runge–Kutta method whose convergence was rigorously proved in [149]) and then apply the Hilbert transform (15) in accordance with well-developed methods [110].

The time dependence of the instantaneous phase difference calculated using the concept of analytic signal is shown in Fig. 12a for various values of noise intensity D. One can see that there exists a certain optimal

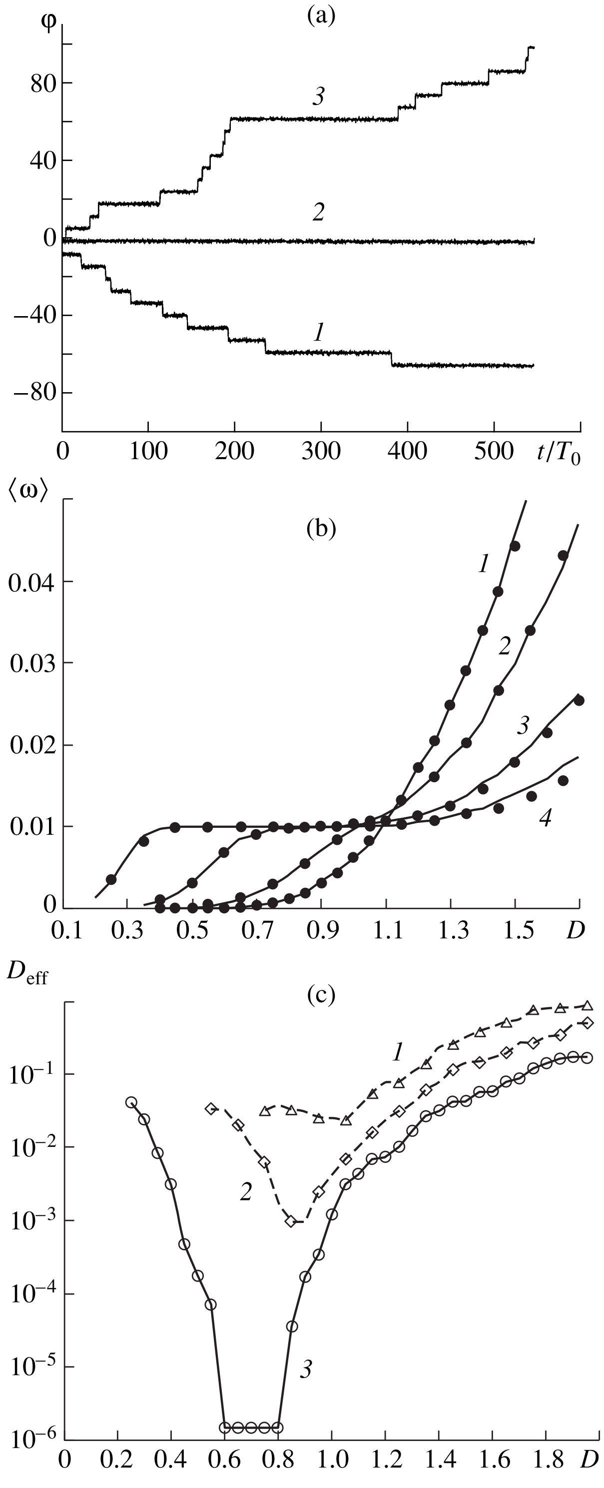

Fig. 12. Phase synchronization of stochastic switchings in system (49); (a) instantaneous phase difference for variations in the noise intensity D: (*1*) $D = 0.44$, (*2*) $D = 0.80$, and (*3*) $D = 1.05$; (b) mean frequency (19) (solid lines) and mean switching frequency (21) (dots) versus noise intensity for various amplitudes A of the external force: (*1*) $A = 0$, (*2*) $A = 1$, (*3*) $A = 2$, and (*4*) $A = 3$; and (c) effective diffusion coefficient as a function of noise intensity for various amplitudes A of the external force: (*1*) $A = 1$, (*2*) $A = 2$, and (*3*) $A = 3$. The parameters of the system are as follows: $\alpha = 5$, $\beta = 1$, and $\Omega_0 = 0.01$.

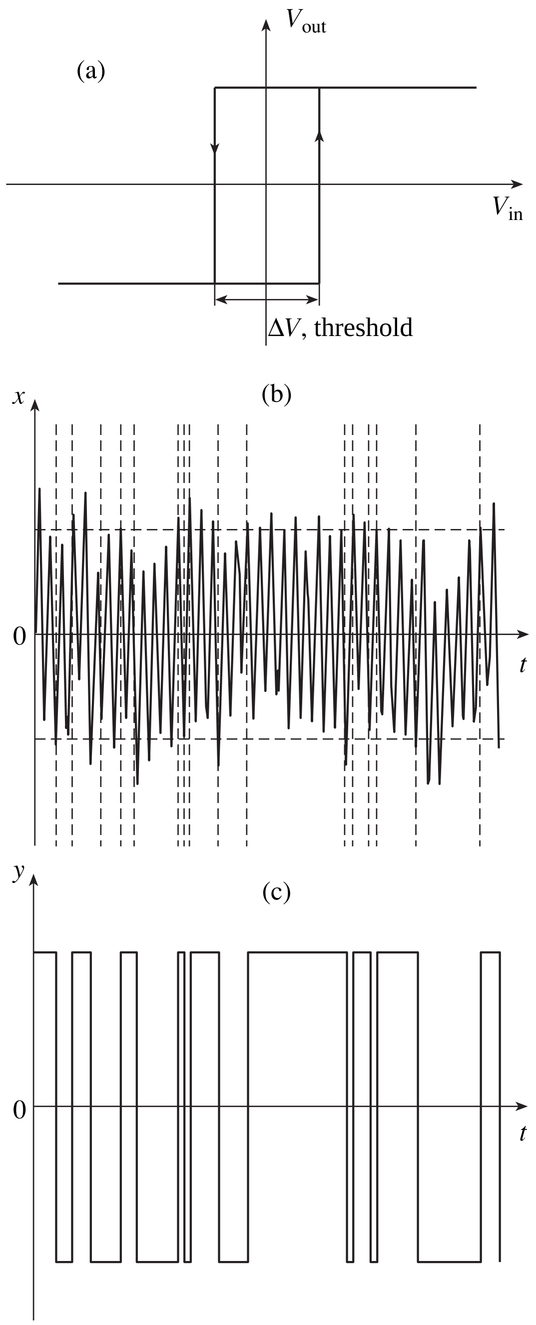

Fig. 13. (a) The characteristic of the Schmitt trigger; (b) the input signal $x(t) = s(t) + \eta(t)$ of the Schmitt trigger, where $s(t) = 0.25\sin(0.1t)$ and $\eta(t)$ is colored noise with a correlation time of τ_c = 0.1; and (c) the output signal $y(t)$. The dashed lines indicate the switching moments of the trigger.

noise intensity such that the instantaneous phases of the response and the external signal prove to be locked during a sufficiently long period of time. When D deviates from this optimal value, the phase difference starts to grow making 2π jumps. Figure 12b represents the mean switching frequency versus the noise intensity for various amplitudes of the modulation signal, calculated with the use of definitions (14)–(18) and definition (21) [38, 94, 165].

These results clearly demonstrate the locking of the mean switching frequency, which was first established in [93]. Figure 12b shows that there exits a synchronization threshold and that the domain of synchronization increases with the modulation amplitude. In the absence of an external signal (A = 0), the mean frequency monotonically increases in accordance with Kramers law (45). When $A \geq 1$, the graph of $\langle\omega\rangle$ versus D shows a feebly marked kink; when A = 2, one can clearly see a plateau where $\langle\omega\rangle$ does not depend on D. A further increase in the amplitude, A = 3, leads to the expansion of the synchronization domain.

In accordance with the definition of effective synchronization (see Section 1), we have to calculate the effective diffusion coefficient D_{eff}. The results of calculations are presented in Fig. 12c. One can see that the diagram of D_{eff} versus noise intensity has a minimum that is more pronounced the larger the amplitude of the external signal. The quantitative values of D_{eff} in the synchronization domain show that the frequency and phase locking phenomena occur at times that are greater more than by a factor of 10^3 than the period of external force. One can state the existence of effective synchronization on the fundamental mode, which manifests itself in the locking of the phase and frequency of switchings by external signals. A nontrivial fact is that the introduction of additional noise into a system results in the ordering of the phase dynamics of the system.

Phase Synchronization of the Switchings of the Schmitt Trigger: Physical Experiment

The forced synchronization of noise-induced switchings can clearly be illustrated by an example of the simplest bistable system. Such a system is represented by the Schmitt trigger perturbed by an external periodic signal and noise [150, 151]. The dynamics of the Schmitt trigger only involves random transitions between two fixed states and, therefore, can be interpreted as the model for the global dynamics of the overdamped oscillator (44), that takes into account oscillations within potential wells. The characteristic of the Schmitt trigger and the signals at its input and output are shown in Figs. 13a–13c.

Experiments on the measurement of the mean switching frequency for a Schmitt trigger were carried out in [93]. A noise signal with a cutoff frequency of f_c = 100 kHz and a periodic signal of frequency f_0 = 100 Hz were applied to the Schmitt trigger with a threshold of ΔV = 150 mV. The mean frequency was calculated by formula (20) using the output signal, which represents a telegraph process and is recorded in a computer. The results of measurements are shown in Fig. 14a. When the signal is weak, the dependence of the mean frequency on the intensity follows Kramers law. As the signal amplitude increases, the dependence of the mean frequency on the noise intensity becomes qualitatively different: a range of values of the noise intensity appears where the mean frequency is virtually independent of noise and remains equal to the signal frequency to within experimental error. One can observe the locking of the mean frequency of switchings [93]. Repeatedly measuring the mean frequency for various values of the signal amplitude and frequency, one can obtain synchronization domains on the plane of parameters "noise intensity–signal amplitude," where the mean frequency is equal to the signal frequency. The domains of synchronization that resemble the Arnold tongues are shown in Fig. 14b. One can see that there exists a threshold value of the signal amplitude A_{th}, starting from which the mean frequency is locked. Upon reaching the threshold, a periodic signal starts to efficiently control the stochastic process of transitions. As the signal frequency increases, the synchronization domains become narrower, while the threshold values of the signal amplitude increase.

The experimental results considered can easily be reproduced by numerical simulation. The Schmitt trigger is described by the following relation [150]:

$$y(t+\Delta t) = \operatorname{sgn}[Ky(t) - \xi(t) - A\sin\Omega t]. \qquad (51)$$

The parameter $K = 0.2$ characterizes the threshold of the trigger, and $\xi(t)$ is exponentially correlated noise with a correlation time of $\tau_c = 10^{-2}$ and intensity D, that is described by the Ornstein–Uhlenbeck process

$$\dot{\xi} = -\frac{1}{\tau_c}\xi + \frac{\sqrt{2D}}{\tau_c}w(t), \quad \langle w(t)w(t+\tau)\rangle = \delta(\tau), \qquad (52)$$

where $w(t)$ is white noise.

The dependence of the mean switching frequency of the trigger on the noise intensity, obtained by the numerical simulation of Eqs. (51) and (52) for various amplitudes of the modulation signal, completely agrees with the experimental data presented in Fig. 14.

Let us note two important moments. Although the signal amplitude is finite, it remains so small that there are no switchings in the system in the absence of noise. Thus, noise is a necessary component of the phenomenon considered, which is accompanied by the locking of the instantaneous phase and the mean frequency of a random oscillatory process of switchings. Moreover, in the synchronization regime, the level of phase fluctuations is substantially reduced: the diffusion coefficient of the instantaneous phase difference becomes minimal.

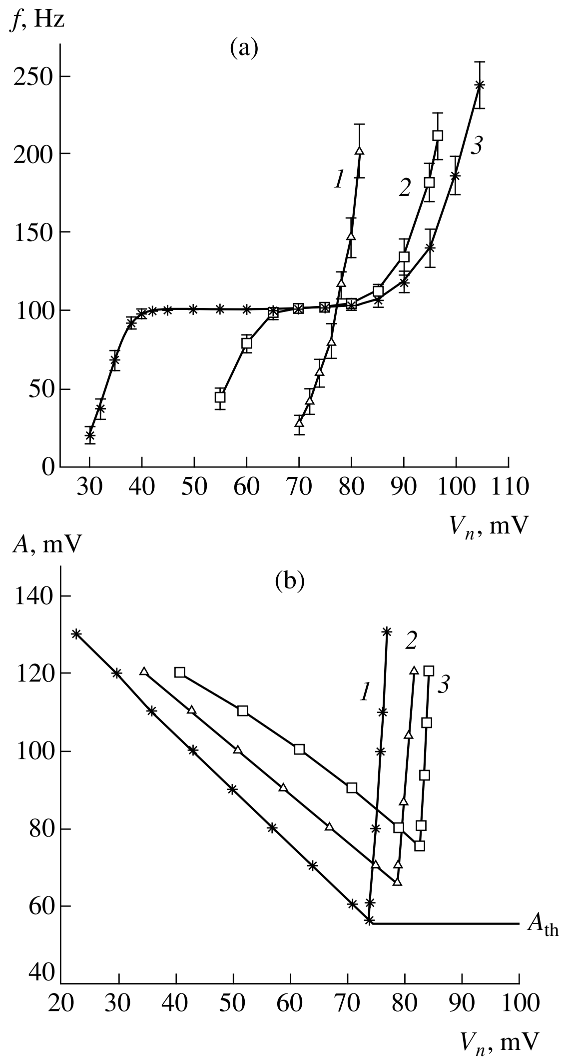

Fig. 14. (a) Experimental dependence of the mean frequency of the output signal of the Schmitt trigger on the noise amplitude V_n for various amplitudes A of signal: (*1*) A = 0 mV, (*2*) A = 60 mV, and (*3*) A = 100 mV and (b) synchronization domains of the Schmitt trigger for various frequencies of the signal: (*1*) f_0 = 100 Hz, (*2*) f_0 = 250 Hz, and (*3*) f_0 = 500 Hz.

Synchronization of Switchings in Chaotic Systems

The systems exhibiting dynamical chaos are characterized by the coexistence of various types of attractors in the phase space [112]. In the absence of external noise, the phase trajectory belongs to one or another attractor, depending on initial conditions. The influence of external noise gives rise to random switchings between the coexisting attractors of the system; the statistics of these switchings is determined by the properties of noise and the dynamical system.

The theoretical analysis of the effect of external noise on the regimes of dynamical chaos can be carried out in the limit of low [152–154] and high [155] Gaussian noise. The theory of small random perturbations of dynamical systems [156] is based on the concept of quasipotential and has recently been extended to systems with complex dynamics [157, 158].

Indeed, after eliminating the intrawell dynamics (motions on chaotic attractors) and restricting the anal-

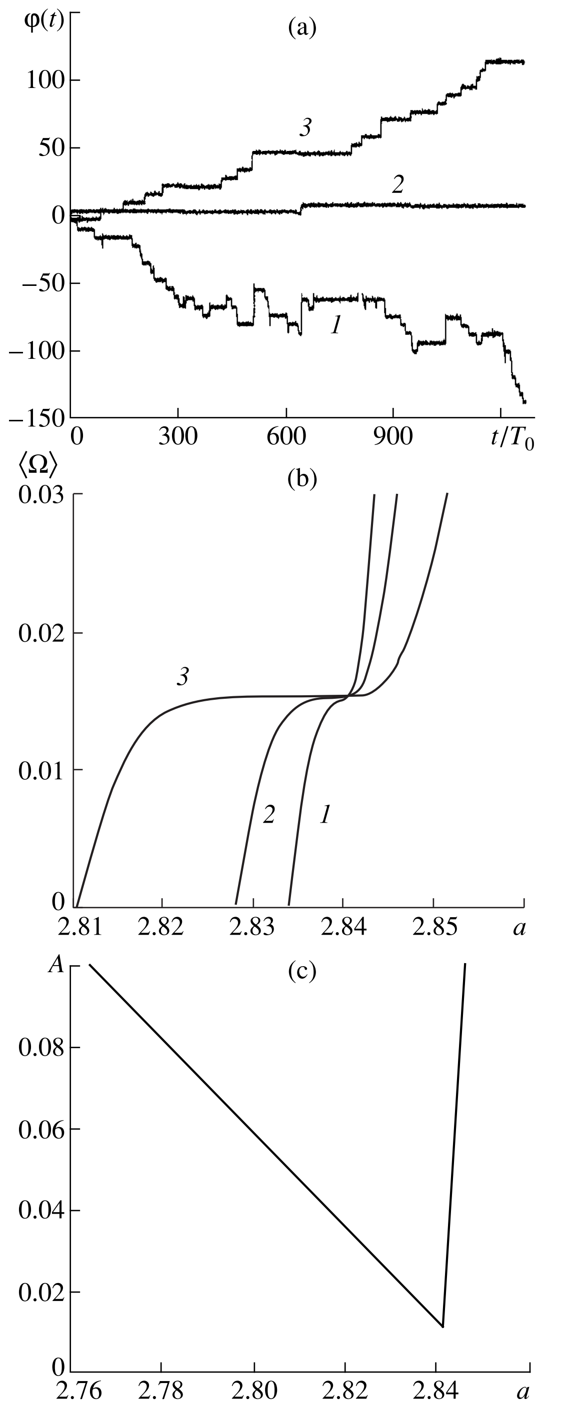

Fig. 15. (a) Instantaneous phase difference as a function of time for various values of the control parameter a: (*1*) a = 2.821, (*2*) a = 2.835, and (*3*) a = 2.8435; (b) mean switching frequency as a function of parameter a for various signal amplitudes A: (*1*) A = 0.005, (*2*) A = 0.025, and (*3*) A = 0.01; and (c) synchronization domain of switching frequency for map (54).

ysis to a slow process of noise-induced switchings, the synchronization problem is qualitatively reduced to the synchronization of switchings of the Schmitt trigger. This fact was experimentally confirmed in [95, 96, 160].

Of interest is the problem of the synchronization of switchings in chaotic systems that can be realized in deterministic systems (in the absence of noise).

It is known that the crises of attractors occur under the variation of the control parameters of chaotic systems. The union of two attractors giving rise to the dynamic intermittency of chaos–chaos type [161], when the phase trajectory stays for a long period of time on each of the joined attractors and performs irregular transitions between them, may serve as an example of a crisis. Note that such random switchings occur in the absence of external noise and are controlled by a deterministic law [162, 163]. For systems with the chaos–chaos-type intermittency, the average time T_i during which the phase trajectory stays on the attractor obeys universal laws of the type [95, 160, 161, 163]:

$$T_i \propto (a - a_{\mathrm{cr}})^{\nu}, \tag{53}$$

where a is a parameter of the system, a_{cr} is the threshold bifurcation value of the parameter under which a crisis and intermittency arise, and ν is a universal constant. Thus, the role of the noise intensity is played here by the parameter of the system that controls a slow time scale. Applying a periodic signal to this system and changing the control parameter, one can achieve a situation when the transitions between the united attractors will be synchronized.

Let us illustrate the aforesaid by a simple example of the discrete system

$$x_{n+1} = (ax_n - x_n^3)\exp\left(\frac{-x_n^2}{b}\right) + A\sin(\Omega_0 n). \tag{54}$$

System (54) represents a one-dimensional cubic map perturbed by a weak periodic signal ($A \ll 1$). The exponential coefficient is introduced to prevent the trajectory from going to infinity. Consider the properties of the map in the absence of perturbation ($A = 0$).

When $a < a_{\mathrm{cr}} = 2.839\ldots$, two chaotic attractors coexist in the system that are separated by a saddle point $x_n = 0$. When the parameter reaches the critical value a_{cr}, a crisis is realized, and the attractors are united, giving rise to the chaos–chaos-type dynamical intermittency.

Consider the reaction of system (54) to a periodic perturbation when the parameter a exceeds the critical value, $a > a_{\mathrm{cr}}$. According to the results of calculations, the mean switching frequency monotonically increases with the parameter a; i.e., the parameter efficiently controls the mean period of aperiodic oscillations of the system.

The locking of the mean switching frequency in system (54) by an external periodic signal is illustrated in Fig. 15. As the signal amplitude A increases, the synchronization domain with respect to the parameter a increases, as is to be expected (Figs. 15a and 15b). Figure 15c shows that this domain represents a typical synchronization domain, just as in the case of the Schmitt trigger (Fig. 14). The synchronization phenomenon also has a threshold character. Thus, the external synchronization of the switching frequency in a deterministic chaotic system is reliably recorded and proves to be qualitatively equivalent to the noise-induced stochastic synchronization. The investigations have shown that the locking of the switching frequency is a universal phenomenon and manifests itself in a wide class of dynamical systems with an intermittency regime [94–97, 164, 165].

Stochastic Synchronization of Bistable Systems by an External Chaotic Signal

The conception of phase synchronization based on the notion of instantaneous phase of a nonperiodic signal (14), (16) allows one to make substantial progress toward the generalization of the classical ideas of synchronization. For example, consider a certain bistable oscillator subject to an irregular oscillatory signal. Is it possible, by using noise, to realize the locking of the instantaneous phase and frequency of switchings of the bistable oscillator by an external nonperiodic signal? If yes, what is the physical meaning of the concept of stochastic synchronization in this case? To answer these questions, consider a simple but illustrative example. In contrast to the case of (46), we investigate the switchings of a bistable oscillator when it is subject to a nonperiodic signal [166],

$$\dot{x} = \alpha x - \beta x^3 + \sqrt{2D}\xi(t) + k\mu(t), \qquad (55)$$

where $\xi(t)$ is Gaussian noise of intensity D, $\alpha, \beta > 0$, k is a certain constant, and $\mu(t)$ is the signal $x_1(t)$ of the Rössler chaotic system [166, 167]:

$$\begin{aligned} \dot{x}_1 &= -\tau(x_2 + x_3), \\ \dot{x}_2 &= \tau(x_1 + 0.15x_2), \\ \dot{x}_3 &= \tau[0.2 + x_3(x_1 - c)]. \end{aligned} \qquad (56)$$

To slow down the oscillation process in (56), a scale factor $\tau = 0.01$ is introduced into (56); the parameter $c > 6.1$ is chosen to implement a chaotic regime of oscillations, and $\mu(t) = x_1(t) - 1.025$. It is known that the oscillation $x_1(t)$ in system (56) in the spiral-chaos regime represents a nonperiodic process of the type $x_1(t) \simeq A(t)\sin[\Omega(t)t]$, where $A(t)$ is an aperiodic function of time and $\langle\Omega(t)\rangle = \Omega_0 = 2\pi/T_0$, where T_0 is the time of one turn of the phase trajectory, averaged over the attractor. Let us choose the parameters of system (56) and the coefficient k so that the external signal $k\mu(t)$ cannot overcome the potential barrier in (55) in the absence of noise. Under these conditions, consider the dynamics of system (55) under the influence of noise.

The results of calculations are represented in Fig. 16. The time-dependence of the instantaneous phase difference for an optimal noise level of $D = 0.79$ (Fig. 16a) provides evidence for the regime of effective phase synchronization. The locking of the mean switching frequency of a bistable oscillator by a chaotic signal is illustrated in Fig. 16b. The investigations have shown that the regime of effective synchronization is characterized by the synchronization domain of the type shown in Fig. 14b, within which the effective diffusion coefficient D_{eff} has a pronounced minimum.

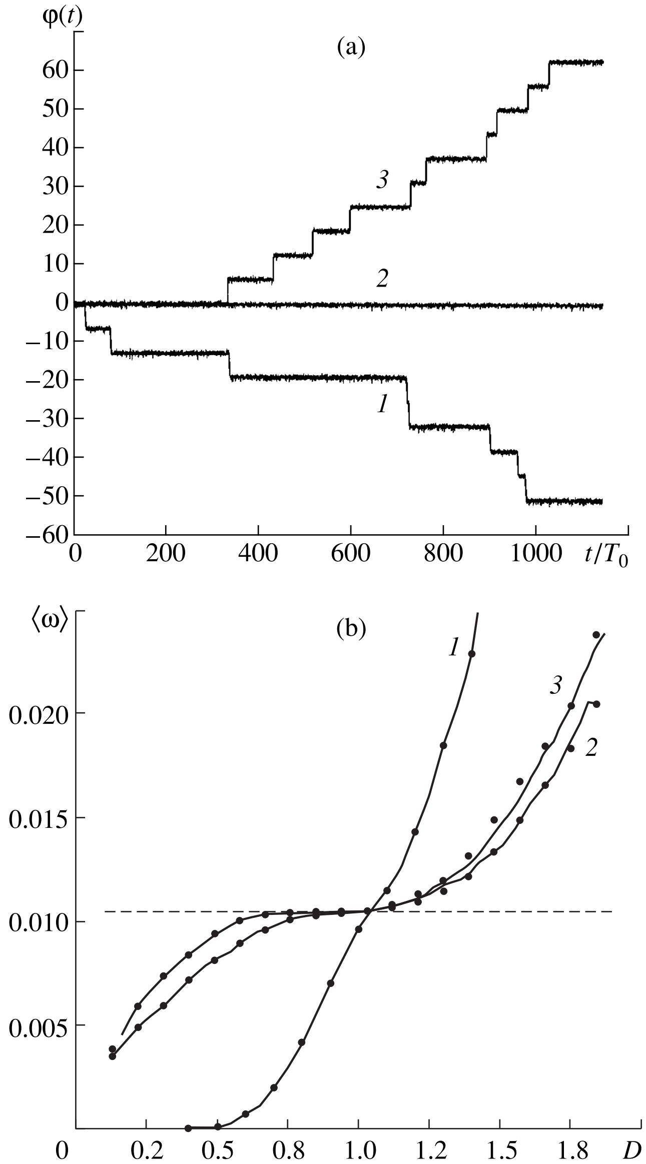

Fig. 16. Stochastic synchronization of bistable system (55) by an external chaotic signal. (a) Time variation of the instantaneous phase difference for $\alpha = 5$; $\beta = 1$; $k = 0.35$, $\mu(t) = x_1(t) - 1.02518$; and (*1*) $D = 0.68$, (*2*) $D = 0.79$, and (*3*) $D = 1.0$. (b) The mean frequency (19) (solid lines) and the mean switching frequency (21) (dots) versus noise intensity for various values of parameters c and k: (*1*) $k = 0$, (*2*) $c = 7.1$ and $k = 0.35$, and (*3*) $c = 8.5$ and $k = 0.3$. The other parameters are as follows: $\alpha = 5$, $\beta = 1$, $\mu(t) = x_1(t) - 1.02518$. The dashed line represents the mean frequency of the external chaotic signal.

The comparative analysis of the processes at the input and output of the bistable system (55) has shown

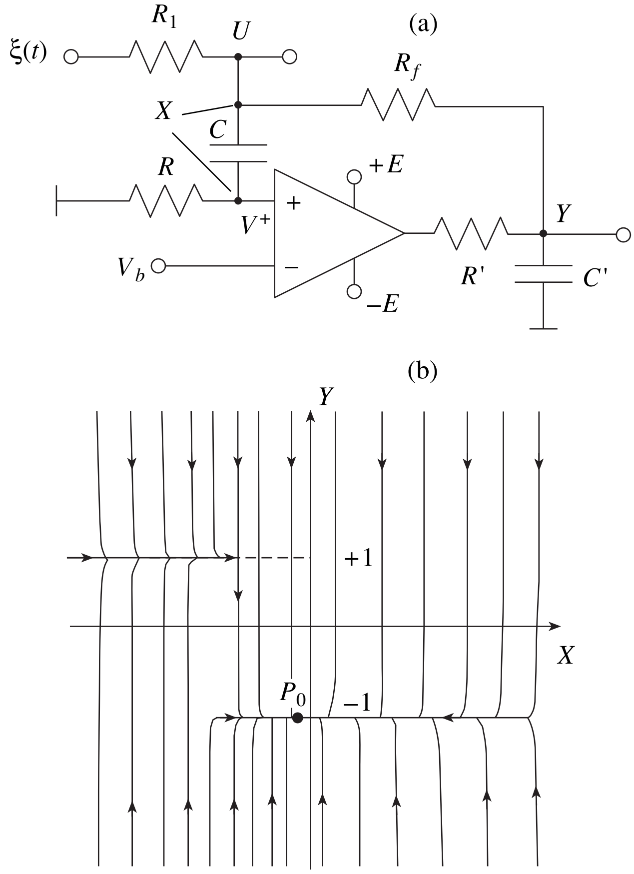

Fig. 17. (a) Schematic diagram of the monovibrator used in a full-scale experiment and (b) basic features of the phase portrait of system (57).

that, in the synchronization regime, the switchings occur simultaneously (on the average) with the maxima of the input signal. Since the sequence of these maxima is irregular, the switchings at the output are also aperiodic. However, the times (and, consequently, the instantaneous phases) of switchings prove to be coherent. This provides a physical explanation of the stochastic phase synchronization by an external aperiodic signal.

The phenomenon described above is fundamentally general for a wide class of systems that are bistable in the generalized sense. In particular, stochastic synchronization of the Kramers oscillator by a signal of the Lorentz system was investigated in [166]. The results of these investigations are completely identical to the results considered above. Undoubtedly, the synchronization of a complex signal is also possible in chaotic bistable systems in which the statistics of switchings depends on a parameter.

Virtually all the above phenomena of synchronization of stochastic oscillations by an external signal also manifest themselves in the interaction of bistable systems. In other words, we have not only an external, but also a mutual, stochastic synchronization. All the characteristic features of external synchronization indicated above qualitatively apply to the case of mutual synchronization [92, 168].

Coherence Resonance in Excitable Systems. Synchronization of a Stochastic Limit Cycle

The stochastic synchronization of the Kramers oscillator yields a striking, but by no means the only, illustration of the increasing degree of order of oscillations in the presence of noise. Another example of similar nonlinear stochastic phenomena can be observed in the so-called excitable systems. These systems are characterized by the generation of pulses under the conditions when the external signal exceeds a certain threshold level. The frequency and shape of the pulses of a response are determined both by the external signal and by the parameters of the system. Excitable systems serve as models of a certain type of neurons [169–171]. They also qualitatively describe the kinetics of certain chemical reactions and the phenomena in Josephson junctions [172, 173]. The phenomenon of *coherence resonance* was established for excitable systems. In this situation, the degree of order increases when the system is subject to the noise of optimal intensity [174–176]. A noise-induced oscillatory process becomes similar to noisy periodic self-oscillations [177–179]. In view of this, an attempt was made to introduce the concept of a *stochastic limit cycle* and investigate the possibility of its synchronization [98–100]. As an example of this phenomenon, we consider the results of investigating a noisy monovibrator that has recently been carried out by D.E. Postnov [180].

The schematic diagram of a single-shot flip-flop is shown in Fig. 17a. Assuming that $R'/R_f \ll 1$, we can derive the following simplified model equations of the monovibrator (in dimensionless units):

$$\dot{x} = \alpha y - \beta x + A\xi(t), \quad \varepsilon\dot{y} = f(\dot{x} - V) - y, \quad (57)$$

where x and y are normalized voltages across the capacitors C and C'; ε, α, β, V, and A are parameters defined by the elements of the circuit; $\xi(t)$ is external noise of intensity D; and the nonlinearity is defined by the Heaviside function: $f(z) = +1$ if $z > 0$, $f(z) = -1$ if $z < 0$, and $f(z) = 1/2$ if $z = 0$. System (57) has a unique stable node-type equilibrium state with the coordinates $x_0 = (A - \alpha)/\beta$ and $y_0 = -1$. The basic properties of the phase portrait of system (57) are represented in Fig. 17b and are typical of many excitable systems [169, 170, 176]. For a monovibrator, the return time of the trajectory to the equilibrium state from the excited state is defined by the parameters of the circuit, while the activation time (the excitation time) is determined by the properties of the external signal.

Consider a coherence resonance in a monovibrator that was experimentally investigated in [99, 180]. Let us analyze the behavior of the power spectrum of the output signal under the variation of the intensity of external noise. Under low-intensity noise, the monovibrator generates a random sequence of pulses with a relatively low mean frequency. After reaching a certain optimal noise intensity $D \approx D_0$, the pulse sequence becomes more regular. The mean duration of pulses approaches the mean pulse spacing. In this case, the output signal becomes similar to a periodic one. As the noise intensity $D > D_0$ increases, the mean pulse spacing appreciably decreases. The output signal again appears to be completely random. Figure 18 represents the power spectra of the signals at the output of the monovibrator for three characteristic values of the noise intensity. One can see that, for an optimal noise intensity (curve *2*), the power spectrum of the output signal has a clear-cut peak, which corresponds to the regime of oscillations close to periodic ones.

The degree of regularity of the output signal can be characterized by various methods [174–176, 180]. However, from the radiophysical point of view, it suffices to have the results of measurements of the spectrum (Fig. 18). It is beyond any doubt that the degree of regularity for the optimal noise level $D \approx D_0$ (curve *2*) is greater than that for $D < D_0$ (curve *1*) and $D > D_0$ (curve *3*).

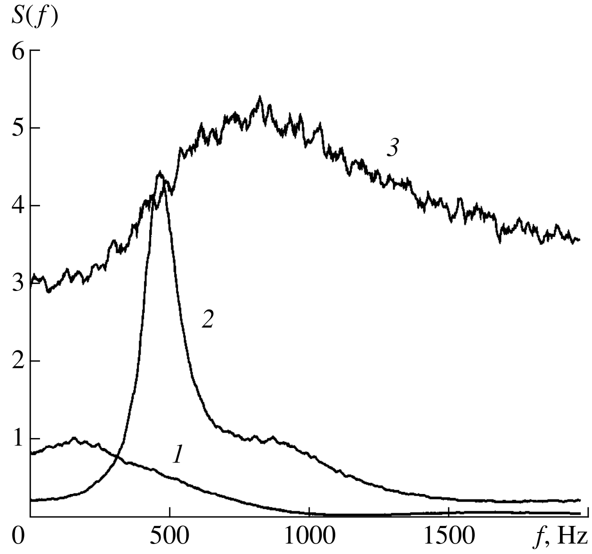

Fig. 18. Typical power spectra of the output signal measured in a radiophysical experiment: (*1*) $D = 15\ \mathrm{mV}^2$, (*2*) $D = 100\ \mathrm{mV}^2$, and (*3*) $D = 600\ \mathrm{mV}^2$.

As we have found out, the response of a monovibrator in the coherent-resonance mode to noise represents a process similar to periodic oscillations in the presence of fluctuations. One could suppose that the flip-flop exhibits the properties of a nonlinear filter whose frequency response essentially depends on the noise intensity. However, this is not quite so. Under the influence of noise, the monovibrator displays the properties of an active system. This fact allows one to characterize the oscillations in the monovibrator for $D \approx D_0$ as a stochastic limit cycle. One can introduce a natural frequency of a stochastic cycle as the frequency corresponding to the peak in the power spectrum. If the monovibrator actually behaves as a self-oscillating system possessing a limit cycle, it must exhibit external and mutual synchronization. The experiments confirmed the presence of such phenomena. As an example, let us consider the results of a radiophysical experiment on the investigation of mutual synchronization of two coupled monovibrators [99, 180]. The results presented in Fig. 19a clearly demonstrate the frequency locking as the coupling coefficient g increases. Moreover, a mismatch between the noise intensities in two monovibrators confirms the robustness of the phenomenon observed. The experiments provide reliable evidence for the fact that there exists a region of locking of the natural frequencies of the partial monovibrators (Fig. 19b). Introducing the instantaneous phase by formula (20), one can observe the phase locking. Similar synchronization phenomena were also observed in other noise-affected excitable systems [98, 100].

Since oscillations in excitable systems are induced by noise and, strictly speaking, represent stochastic processes, the synchronization of these systems, just as the synchronization of noise-affected bistable systems, can be called a stochastic synchronization [94]. The stochastic synchronization in different systems has a common feature: both in the case of bistable systems and excitable systems, the only possible synchronization of noise-induced oscillations is the effective phase synchronization. However, there is an essential difference. For instance, excitable systems similar to the monovibrator considered above are characterized by a clear-cut peak at a certain frequency in the spectrum, whereas the spectrum of stochastic oscillations of bistable systems does not contain such peaks. In addition, in contrast to bistable systems, one could not find out a threshold value of the coupling parameter or the amplitude of a harmonic signal in excitable systems, such that there is no synchronization below this threshold. The stochastic synchronization of excitable systems represents a new complex physical phenomenon that requires further detailed investigation.

4. SYNCHRONIZATION OF CARDIAC RHYTHM

The Phase Synchronization of Cardiac Rhythm by an External Periodic Signal

The cardiovascular system (CVS) of a human is an example of one of the most complex nonlinear oscillating systems. An obvious fact that there exist undamped self-sustained vibrations in a CVS provide evidence for the self-oscillating character of its operation. At the same time, the mechanism of self-oscillations in the CVS has not yet been ultimately revealed. It is known that the cardiac vibrations of a human (or animals) are not strictly periodic [173, 181, 182]. This fact can be attributed to the effect of various noise sources on the

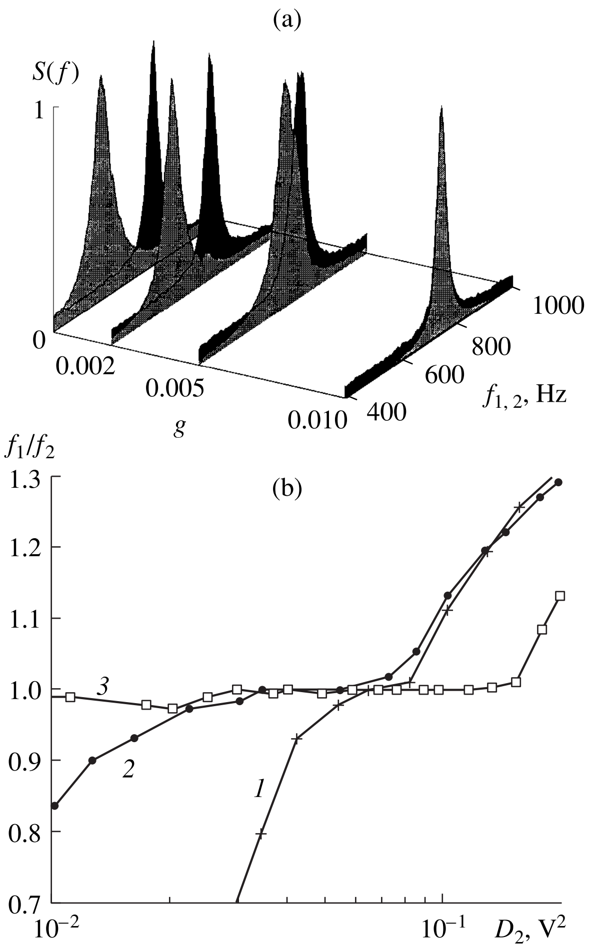

Fig. 19. The locking of the base frequencies f_1 and f_2 corresponding to the maxima in the power spectral density of stochastic oscillations of two coupled monovibrators (radiophysical experiment). (a) Evolution of the power spectra of partial systems under the variation of the coupling parameter g, and (b) frequency ratio f_1/f_2 as a function of the noise intensity D_2 of the second monovibrator for various values of the coupling parameter g: (*1*) $g = 0.01$, (*2*) $g = 0.0125$, and (*3*) $g = 0.03$; the noise intensity of the first flip flop is $D_1 = 0.086$ V^2.

CVS, similar to the case of self-oscillations in a noisy Van der Pol oscillator considered in Section 1. However, there are grounds to suggest that the aperiodic character of cardiac vibrations is associated with the essentially chaotic dynamics of the CVS [173, 182, 183]. Moreover, one should not neglect a self-consistent influence of various subsystems of the organism on the CVS, which complicates the analysis and requires the application of the theory of nonautonomous oscillations.

The following experiment was proposed in [104, 184] to study the external synchronization of the cardiac rhythm. A patient was subject to a weak signal representing a periodic train of pulses with frequency f_F close to the mean pulse rate $\langle f_H \rangle$ of the patient and with duration amounting to $\simeq 10\%$ of the mean duration of one cardiac interval. These pulses were transformed into a synchronous periodic sequence of bursts on a display (a bright red square popped up on the screen) accompanied by a feeble audio signal generated by the computer dynamic speaker.

The synchronization phenomena were investigated by the numerical processing of the electrocardiogram (ECG) signal. The ECGs of a patient were recorded both in the absence and in the presence of an external signal. The ECGs were recorded during 300–600 s in real time. The processing of experimental data involved the calculation of the instantaneous phase difference between the external signal and the ECG of a patient in the steady state in the presence of an external signal. Discrete sequences of *RR* intervals of the ECG were used in calculations to determine the instantaneous phase by formula (19). We recorded 40 ECG signals in 16 young people without signs of CVS diseases. The reliability of the results was tested by special methods [104].

Figure 20a demonstrates typical time-dependence of the instantaneous phase difference normalized by 2π for the case when the mean pulse rate of a patient and the frequency of the external signal differ by 3%. These results provide evidence for the effective phase synchronization of a cardiac rhythm: the instantaneous phase difference remains bounded and close to zero or a multiple of 2π during a time interval of $\Delta t \simeq 150$ s (about 150 cardiac intervals). The variations of the driving frequency (its increase or decrease relative to the mean pulse rate of a patient in the absence of external signal) have allowed us to determine experimentally the frequency-locking region. The results (Fig. 20b) show that the mean frequency of the cardiac rhythm proves to be locked by the external signal within a frequency band of $\pm 5\%$ of the mean frequency of the cardiac rhythm in the absence of an external signal. The calculations have shown that, in the synchronization domain, the effective diffusion coefficient D_{eff} (11) has a clear-cut minimum close to 10^{-2} and sharply increases as it leaves the synchronization domain.

Phase Synchronization of a Cardiac Rhythm by an External Aperiodic Signal

It would be interesting to verify if it is possible to realize a phase synchronization similar to that considered in Section 2 for the cardiac rhythm of a human under the influence of a nonperiodic signal. For this purpose, we chose the external signal in the form of a sequence of *RR* intervals of the ECG obtained in the experiments on another patient. Moreover, we chose the pairs of patients such that the mean durations of the *RR* intervals in their ECGs differed by at least 5%. This means the exit from the synchronization domain at the fundamental mode. Below, we consider the results obtained under the conditions when the mean pulse rate of the driving ECG is 1 Hz, while the mean pulse rate of a patient (in the absence of external signal) is 0.85 Hz.

When there is a substantial difference between the mean frequencies of the master and slave oscillatory processes, the phase synchronism condition satisfies the following general relation:

$$\lim_{t \to \infty} |m\Phi_1(t) - n\Phi_2(t)| < M = \text{const}, \qquad (58)$$

where m and n are integers and $\Phi_{1,2}(t)$ are the instantaneous phases of the compared oscillatory processes.

Typical experimental results are shown in Fig. 20c. The diagram in this figure represents the time-dependence of the instantaneous phase obtained for $m = 7$ and $n = 6$. The phase difference is close to zero in the time interval $\Delta t \simeq 50$ s and is no greater than 2π in the interval $\Delta t \simeq 100$ s. One can speak of the effective phase synchronization corresponding to the resonance $m : n = 7 : 6$. The synchronization by an aperiodic signal has been confirmed in 19 of 20 experiments. Note that, from the viewpoint of condition (58), the results of the preceding section are interpreted as the effective synchronization at the fundamental mode under the condition that $m = 1$ and $n = 1$.

Phase Synchronization of a Cardiorespiratory System: Mutual Synchronization

In [185–187], the authors investigated the synchronization of a cardiac rhythm and a respiratory process. From the viewpoint of the theory of oscillations, we can speak of a mutual synchronization of two oscillating subsystems of a human organism: the respiratory and cardiovascular systems. It is known that the respiratory rate of a relatively healthy human is less than the frequency of cardiac contractions by a factor of three–five. Due to the natural links between these subsystems, the respiration and cardiac contractions exhibit synchronization phenomena when one of these frequencies becomes an integral multiple of the other. However, it is worth emphasizing that the authors of [185–187] were the first who set up and experimentally solved the problem of the phase synchronization of a cardiorespiratory system and determined the instantaneous phase on the basis of the theory of an analytic signal.

Setting aside the purpose and the details of the experiments described in [185–187], we mention the results that are the most important for the present survey. First, it was established that a phase synchronization of the cardiorespiratory system of a human occurs that corresponds to complex resonances of 3 : 1, 5 : 2, and 8 : 3. Second, sound evidence was provided for the effective synchronization such that the phase and frequency are locked on finite time intervals that are greater than the mean respiration period (and, naturally, the mean duration of a cardiac interval).

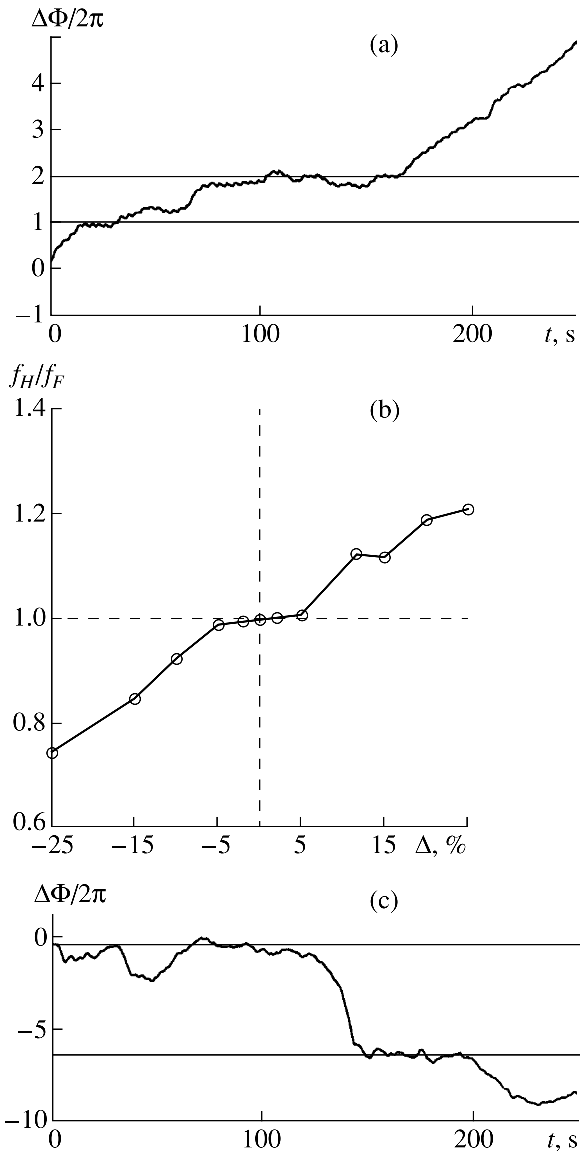

Fig. 20. (a) Instantaneous phase difference between the periodic driving signal and the sequence of *RR* intervals in the ECG of a patient when $\Delta = |\langle f_H \rangle - f_F|/f_F = 3\%$; (b) the frequency ratio f_H/f_F as a function of mismatch Δ manifesting effective phase synchronization; and (c) phase difference between the aperiodic driving signal and the sequence of *RR* intervals in the ECG of a patient for $\Delta = 3\%$.

CONCLUSION

In this survey, we presented the results of investigating synchronization as a fundamentally general phenomenon observed in a wide class of self-oscillating systems generating both periodic and aperiodic oscillations. An attempt was made to consider the synchronization from a unified point of view based on the classical results of the theory of synchronization of periodic oscillations. The synchronization phenomena were investigated as applied to systems that are characterized by a finite number of degrees of freedom in the absence of noise. We did not consider distributed systems, delay systems, as well as their models in the form of chains and arrays of interconnected elements. (The synchronization of oscillatory and wave processes in distributed systems are of special interest, and we are going to write a special paper on this subject.)

Let us briefly summarize the main results. The theory of synchronization of periodic oscillations is presented that involves the problem associated with the effect of fluctuations. Taking into account the fundamental importance of the classical theory for interpreting the results of present-day investigations, as well as the fact that the original classical works on synchronization published by the representatives of the school of L.I. Mandel'shtam, A.A. Andronov, and R.L. Stratonovich are hardly available, we considered this problem in sufficient detail. In particular, we gave a detailed description of the effective synchronization (synchronization in the presence of noise) and focused our attention on the concept of instantaneous phase of an arbitrary oscillatory process.

We discussed the problem of the synchronization of chaotic oscillations generated by nonlinear dynamical systems in the deterministic-chaos regime. We considered the frequency-phase, full, and generalized synchronizations of chaos, which represent a consequent and further development of the concepts of the classical theory. The essential point here is the fact that the synchronization of chaos manifests itself in the locking of the instantaneous phase of nonperiodic oscillations.

We present the results that convincingly demonstrate the possibility of the phase synchronization of stochastic oscillations, i.e., the oscillatory processes whose characteristics are essentially random. In fact, here we deal with a new phenomenon, the stochastic synchronization. At the same time, the results obtained convincingly show that the generalized concepts of the classical theory of phase synchronization are applicable to stochastic oscillating systems.

We also described the synchronization of the cardiac rhythm of a human by periodic and aperiodic signals. The results described are of great interest and importance from the two following points of view. First, they show that the cardiovascular system of a human possesses typical properties of a noisy self-oscillating system and exhibits effective synchronization. Second, the comparison of the results described in Section 4 with those obtained in Sections 1 and 2 shows that the effective synchronization of a CVS is described qualitatively identically irrespective of whether the autonomous oscillating regime in the CVS is periodic or chaotic.

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From the Editors

The account of certain problems given in this survey, especially those concerning the phase synchronization of chaotic and stochastic oscillations, is controversial. Nevertheless, in view of a large number of publications that have been recently devoted to this subject, we consider that it is worthwhile to present this paper in the form submitted by the authors.

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From the Editorial board

Some aspects in the review, especially concerned with phase synchronization of chaotic and stochastic oscillations, can be discussed. However, taking into account a great number of recent publications on this subject, the Editorial board considers that it is expedient to publish this paper in the form presented by the authors.